EEE 324 Digital Signal Processing

Lecture 10

Properties of z-Transform

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z-Transform Properties

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z-Transform Properties

- For the remaining lecture assume the following:

$$x[n] \overset{z}{\leftrightarrow} X(z), \qquad ROC = R_x$$

Here R_x represents a set of values of z such that $r_R < |z| < r_L$

- For two sequences, the z-transform pairs are:

$$x_1[n] \overset{z}{\leftrightarrow} X_1(z), \qquad ROC = R_{x_1}$$

$$x_2[n] \overset{z}{\leftrightarrow} X_2(z), \qquad ROC = R_{x_2}$$

z-Transform Properties

1. **Linearity**

$$a x_1[n] + b x_2[n] \overset{z}{\leftrightarrow} a X_1(z) + b X_2(z), \qquad ROC\ contains\ R_{x_1} \cap R_{x_1}$$

- The case of pole-zero cancellation
  - ROC may be larger than the overlap.
  - E.g., when $x_1[n]$ and $x_2[n]$ are of infinite duration but the linear combination is of finite duration (ROC is the entire z-plane)
- The case of no pole-zero cancellation
  - ROC will be exactly equal to the overlap of the individual ROCs.

z-Transform Properties

2. Time Shifting

$$x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}X(z)$$

$$ROC = R_x(except\ for\ the\ possible\ addition\ or\ deletion\ of\ z = 0\ or\ z = \infty)$$

- n_0 is an integer
  - When it is positive, the sequence $x[n]$ is shifted right
  - When it is negative, the sequence $x[n]$ is shifted left.
- The ROC can change depending on the poles and zeros.
- This property is often useful for obtaining the inverse z-transform.

z-Transform Properties

2. Time Shifting

<u>Example 3.14 (Shifted Exponential Sequence)</u>

z-Transform Properties

3. Multiplication by an Exponential Sequence

$$z_0{}^n x[n] \overset{Z}{\leftrightarrow} X\left(\frac{z}{z_0}\right)$$

$$ROC = |z_0|R_x$$

- The ROC is R_x scaled by $|z_0|$
  - If R_x is the set of values of z such that $r_R < |z| < r_L$,
    - then $|z_0|R_x$ is the set of values of z such that $|z_0|r_R < |z| < r_L|z_0|$
- If $X(z)$ has a pole at $z = z_1$, then $X(z_0{}^{-1}z)$ will have a pole at $z = z_0 z_1$.
- If z_0 is a positive real number,
  - The scaling can be interpreted as a shrinking or expanding of the z-plane i.e., the pole and zero locations change along radial lines in the z-plane.
- If z_0 is complex with unity magnitude, so that $z_0 = e^{j\omega_0}$, the scaling corresponds to a rotation in the z-plane by an angle of ω_0 i.e., the pole and zero locations change in position along circles centred at the origin.
  - Can be interpreted as a Frequency shift or translation, associated with modulation in time domain by complex exponential sequence $e^{j\omega_0 n}$

z-Transform Properties

3. Multiplication by an Exponential Sequence

Example 3.15 (Exponential Multiplication)

z-Transform Properties

4. **Differentiation of $X(z)$**

$$nx[n] \overset{z}{\leftrightarrow} -z\frac{d}{dz}X(z)$$

$$ROC = R_x$$

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z-Transform Properties

4. Differentiation of $X(z)$

Example 3.17 (Second order pole)

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5. Conjugation of a Complex Sequence

$$x^*[n] \overset{Z}{\leftrightarrow} X^*(z^*)$$

$$ROC = R_x$$

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6. Time Reversal

$$x[-n] \overset{Z}{\leftrightarrow} X\left(\frac{1}{z}\right)$$

$$ROC = 1/R_x$$

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z-Transform Properties

6. Time Reversal

Example 3.18

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z-Transform Properties

7. Convolution of Sequences

$$x_1[n] * x_2[n] \overset{z}{\leftrightarrow} X_1(z)X_2(z)$$

$$ROC \text{ contains } R_{x_1} \cap R_{x_2}$$

- The z-transform of the O/P of an LTI system is the product of the z-transform of the I/P and the z-transform of the impulse response.
- The z-transform of the impulse response of an LTI system is typically referred to as the system function.

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z-Transform Properties

7. Convolution of Sequences

Example 3.19 (Convolution using z-transform)

z-Transform Properties

8. Initial Value Theorem

If $x[n]$ is zero for $n < 0$ (i.e., if $x[n]$ is causal), then

$$x[0] = \lim_{z \to \infty} X(z)$$

z-Transform Properties

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Section Reference	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x
	$x_1[n]$	$X_1(z)$	R_{x_1}
	$x_2[n]$	$X_2(z)$	R_{x_2}
3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$
3.4.4	$nx[n]$	$-z \dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
	$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.8	Initial-value theorem:		
	$x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$		

Suggested Reading

Section 3.4 (Oppenheim)

Practice Problems

<u>Problems: 3.2, 3.8, 3.9</u>