EEE 324 Digital Signal Processing

Lectures 11, 12

System Functions for Systems Characterized by LCCDE

Dr. Shadan Khattak

Department of Electrical Engineering

COMSATS Institute of Information Technology - Abbottabad

Contents

Introduction

- From the lecture on z-transform, we know that,

$$Y(z) = H(z)X(z) \qquad (1)$$

- In Eq. (1), $H(z)$ is called the system function.
- Any LTID system is completely characterized by its system function, assuming convergence.
- Both the frequency response and the system function are extremely useful in the analysis and representation of LTI systems.

Frequency Response of LTI Systems

- The frequency response $H(e^{j\omega})$ of an LTI system is defined as the complex gain (eigenvalue) that the system applies to the complex exponential input (eigenfunction) $e^{j\omega n}$.
- We know that,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \qquad (2)$$

where

$X(e^{j\omega})$: DTFT of the input sequence

$Y(e^{j\omega})$: DTFT of the output sequence

Frequency Response of LTI Systems

- The magnitude and phase of the input and output sequences are related as:

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right| . \left|X\left(e^{j\omega}\right)\right| \qquad (3a)$$

$$angle\left(Y\left(e^{j\omega}\right)\right) = angle\left(H\left(e^{j\omega}\right)\right) + angle\left(X\left(e^{j\omega}\right)\right) \qquad (3b)$$

- $\left|H\left(e^{j\omega}\right)\right|$ is called:
  - Magnitude response, or
  - Gain

  of the system
- $angle\left(H\left(e^{j\omega}\right)\right)$ is called:
  - *Phase response, or*
  - *Phase shift*

  of the system

Frequency Response of LTI Systems

- The magnitude and phase effects in Eq. (3a) and Eq. (3b) can be either desirable or undesirable.
- In case they are undesirable, they are called the magnitude and phase distortions respectively.

Frequency Response of LTI Systems

Ideal Frequency Selective Filters

1. The Ideal Low Pass Filter (ILPF)

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \qquad (4)$$

- Selects the low-frequency components of the signal and rejects the high-frequency components.
- Using the formula of inverse DTFT, the impulse response of an ILPF is:

$$h_{lp}[n] = \frac{sin\omega_c n}{\pi n}, \quad -\infty < n < \infty \qquad (5)$$

Frequency Response of LTI Systems

Ideal Frequency Selective Filters

2. The Ideal High Pass Filter (IHPF)

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & \omega_c < |\omega| \leq \pi \end{cases} \qquad (6)$$

Also,

$$H_{hp}\left(e^{j\omega}\right) = 1 - H_{lp}\left(e^{j\omega}\right)$$

And,

$$h_{lp}[n] = \delta[n] - h_{lp}[n] = \delta[n] - \frac{sin\omega_c n}{\pi n} \qquad (7)$$

- The ILPF passes the frequency band $\omega_c < |\omega| \leq \pi$ undistorted and rejects frequencies below ω_c.

Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

Frequency Response of LTI Systems

Ideal Frequency Selective Filters

Issues:

- The ILPF is non-causal and its impulse response extends from $-\infty$ to ∞.
- The ILPF is not computationally realizable.
- The phase response of an ILPF is 0.
- Later, we will see that causal approximations to ideal frequency-selective filters must have a non-zero phase response.

Frequency Response of LTI Systems

Phase Distortion and Delay

- For an ideal delay system $y[n] = x[n - n_d]$,
  - the impulse response is: $h_{id}[n] = \delta[n - n_d]$, and
  - the frequency response is: $H_{id}\left(e^{j\omega}\right) = e^{-j\omega n_d}$
  - the magnitude response is: $\left|H_{id}\left(e^{j\omega}\right)\right| = 1$
  - the phase response is: $angle\left(H_{id}\left(e^{j\omega}\right)\right) = -\omega n_d, \quad |\omega| < \pi$

Frequency Response of LTI Systems

Phase Distortion and Delay

- Generally, while designing systems, the aim is to have a linear phase response rather than a zero phase response.
- E.g., an ILPF with linear phase is defined as:

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \quad (8)$$

With an impulse response

$$h_{lp}[n] = \frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}, \quad -\infty < n < \infty \quad (9)$$

- These filters do two things:
  - Isolate a band of frequencies
  - Delay the output by n_d

Frequency Response of LTI Systems

Phase Distortion and Delay

Group Delay:

- Is a measure of linearity of the phase.
- Relates to the effect of the phase on a narrowband signal.

$$angle\left(H\left(e^{j\omega}\right)\right) \cong -\emptyset_0 - \omega n_d$$

$$\tau(\omega) = grd\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right\}\right.$$

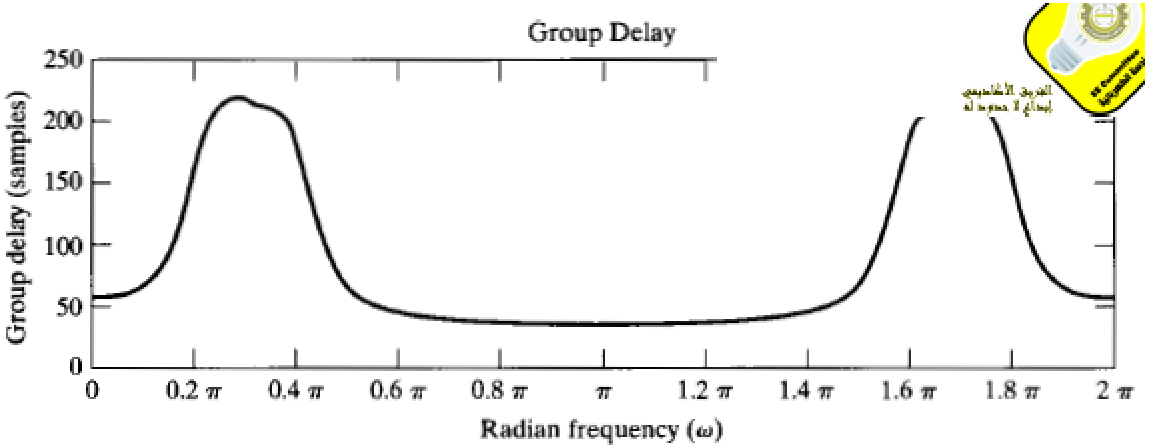
- The deviation of the group delay from a constant indicates the degree of nonlinearity of the phase.

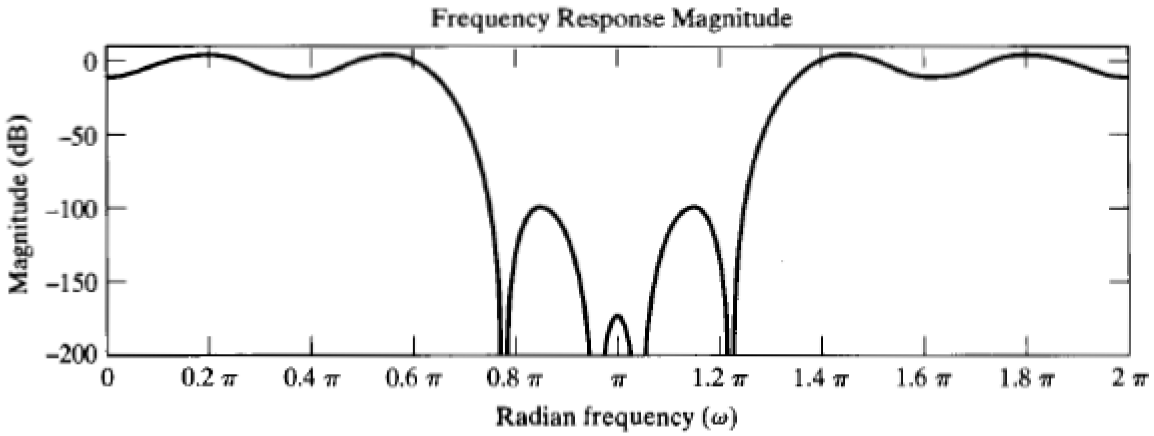
Frequency Response of LTI Systems

<u>Phase Distortion and Delay</u>

Group Delay:

Example 5.1 (Effects of Attenuation and Group Delay)





Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

Frequency Response of LTI Systems

Phase Distortion and Delay

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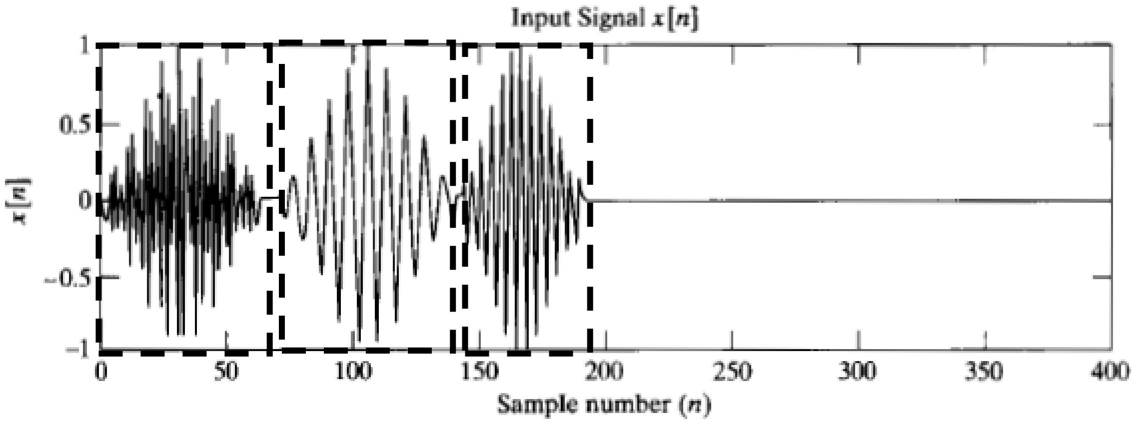

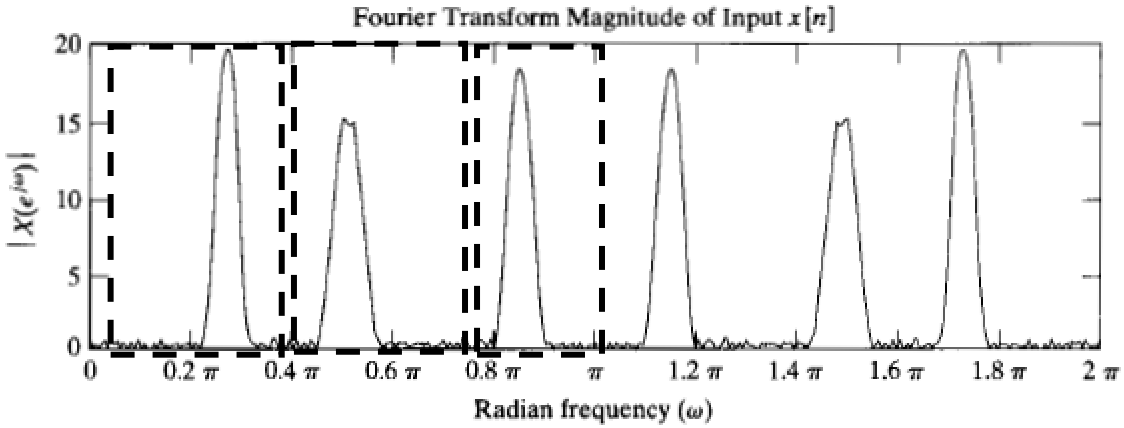

Three narrowband pulses: $\omega = 0.85\pi, \omega = 0.25\pi, \omega = 0.5\pi$

Frequency Response of LTI Systems

Phase Distortion and Delay

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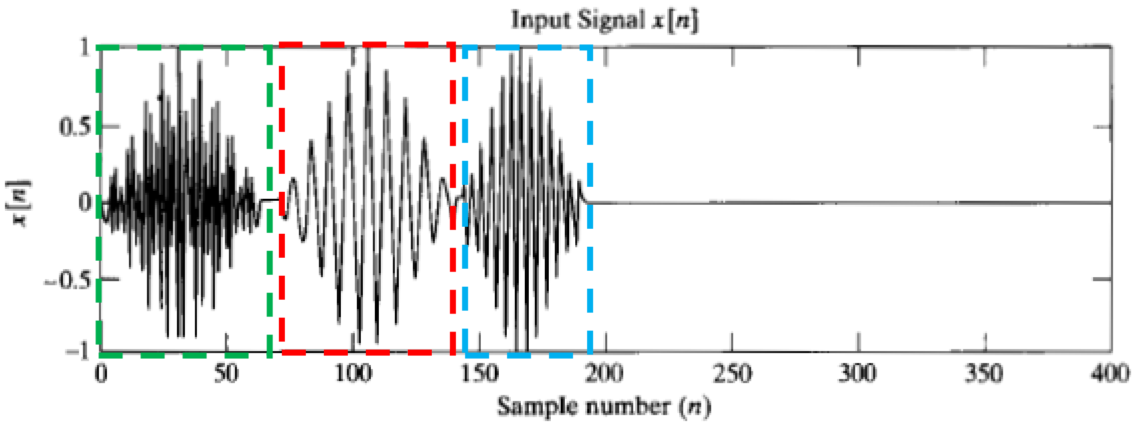

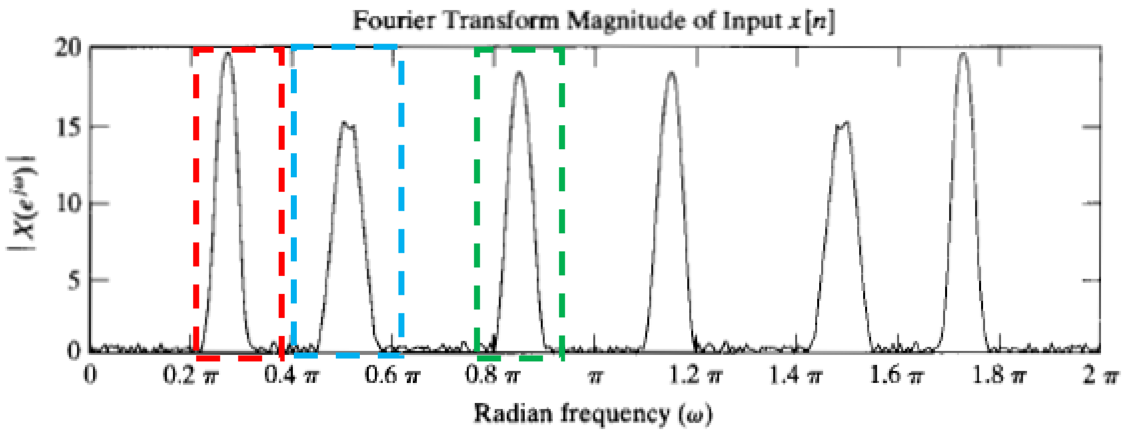

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Frequency Response of LTI Systems

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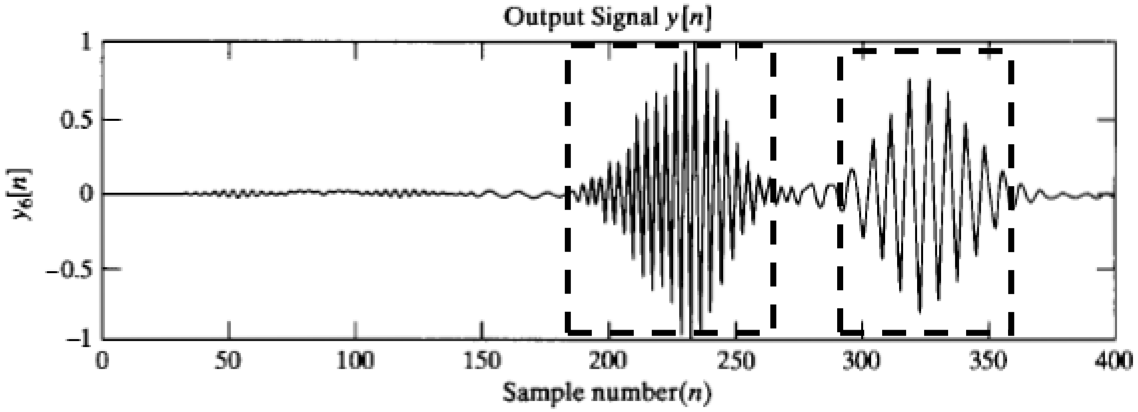

Frequency Response of LTI Systems

Phase Distortion and Delay

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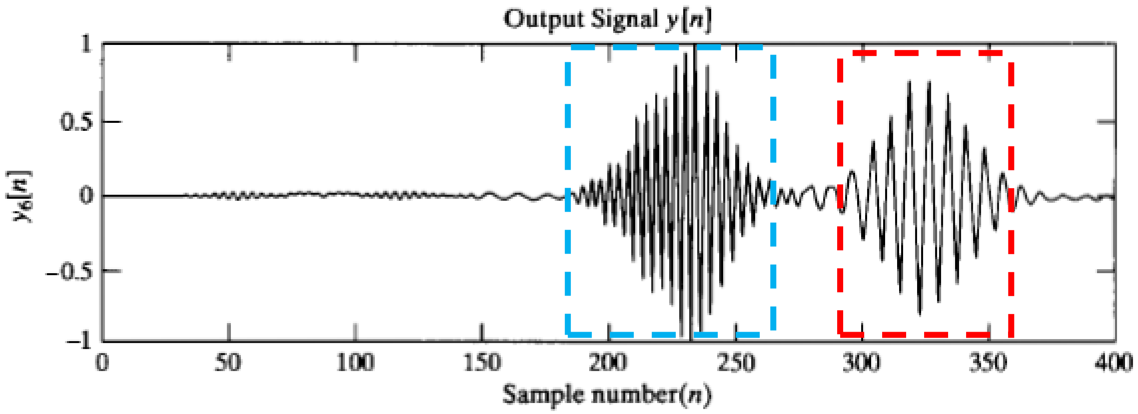

Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

System Functions for Systems Characterized by LCCDE

- While ideal FS filters are useful conceptually, they cannot be implemented with finite computations.
  - Therefore, we consider a class of systems that can be implemented as approximations to ideal FS filters.
- A class of systems whose i/p and o/p satisfy an LCCDE of the following form are important.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (10)$$

- If the auxiliary conditions correspond to initial rest, the system will be:
  - Causal
  - Linear
  - Time invariant

System Functions for Systems Characterized by LCCDE

- The properties and characteristics of LTI systems for which the i/p and o/p satisfy an LCCDE are best developed through the z-transform.
- Applying the z-transform to both sides of Eq. (10), and using time shifting and linearity properties, we obtain,

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

or,

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z) \qquad (11)$$

System Functions for Systems Characterized by LCCDE

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad (12)$$

Or, in factored form,

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad (13)$$

- Each of the factor $(1 - c_k z^{-1})$ contributes a zero at $z = c_k$ and a pole at $z = 0$.
- Each of the factor $(1 - d_k z^{-1})$ contributes a pole at $z = d_k$ and a zero at $z = 0$.

System Functions for Systems Characterized by LCCDE

<u>Example 5.2 (Second Order System)</u>

Transfer function to Difference Equation

Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

System Functions for Systems Characterized by LCCDE

Stability and Causality

- ***Stability*** implies that:
  - ROC must include the unit circle.
  - Impulse response must be absolutely summable.
- ***Causality*** implies that:
  - ROC must be outward of the outermost pole.
  - Impulse response must be right right-sided.
- ***For a system to be both stable and Causal,***
  - All the poles of the system must lie within the unit circle.

System Functions for Systems Characterized by LCCDE

<u>Example 5.3 (Determining the ROC)</u>

System Functions for Systems Characterized by LCCDE

Inverse Systems

- A system with system function $H_i(z)$ such that if it is cascaded with $H(z)$, the overall effective system function is unity; i.e.,

$$G(z) = H(z)H_i(z) = 1 \qquad (14)$$

It implies that

$$H_i(z) = \frac{1}{H(z)} \qquad (15)$$

The equivalent time-domain condition is:

$$g[n] = h[n] * h_i[n] = \delta[n] \qquad (16)$$

The frequency response of the inverse system, if it exists, is

$$H_i\left(e^{j\omega}\right) = \frac{1}{H\left(e^{j\omega}\right)} \qquad (17)$$

System Functions for Systems Characterized by LCCDE

Inverse Systems

- The log magnitude, phase, and group delay of the inverse system are negatives of the corresponding functions for the original system.
- Not all systems have an inverse. E.g.
  - The ILPF does not have an inverse.
- The class of systems with rational system functions is an example of systems which do have inverses.
- The poles of $H_i(z)$ are the zeros of $H(z)$ and vice versa.
- What about the ROC of $H_i(z)$?
  - The ROC of $H(z)$ and $H_i(z)$ must overlap.

System Functions for Systems Characterized by LCCDE

Inverse Systems

- If $H(z)$ is causal, its ROC is $|z| > \max_k |d_k|$
  - So, any appropriate ROC for $H_i(z)$ that overlaps with this region is a valid ROC for $H_i(z)$.

System Functions for Systems Characterized by LCCDE

Inverse Systems

Example 5.4 (Inverse System for First-Order System):

Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

System Functions for Systems Characterized by LCCDE

Inverse Systems

Example 5.5 (Inverse For System with a Zero in the ROC)

Two valid inverse systems

Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

System Functions for Systems Characterized by LCCDE

Inverse Systems

Summary of Ex. 5.4, 5.5

- If $H(z)$ is a causal system with zeros at $c_k, k = 1, \ldots, M$, then its inverse system will be causal if and only if we associate the ROC$|z| > \max_k |c_k|$ with $H_i(z)$.
- If we also require that the inverse system be stable, then the ROC of $H_i(z)$ must include the unit circle. Therefore, it must be true that

$$\max_k |c_k| < 1$$

  - All the zeros of $H(z)$ must be inside the unit circle.
  - An LTI system is stable and causal and also has a stable and causal inverse if and only if both the poles and the zeros of $H(z)$ are inside the unit circle.
  - Such systems are called ***minimum phase systems***.

System Functions for Systems Characterized by LCCDE

Impulse Response for Rational System Functions

We know that for a system with only first order poles,

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \qquad (18)$$

If the system is assumed to be causal, then,

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k (d_k)^n u[n]$$

System Functions for Systems Characterized by LCCDE

<u>Impulse Response for Rational System Functions</u>

Based on the nature of the impulse response, two important classes of LTI systems exist:

1. ***<u>Infinite Impulse Response (IIR)</u>***
   1. At least one non-zero pole of $H(z)$ is not cancelled by a zero.
   2. At least one term of the form $A_k(d_k)^n u[n]$
   3. h[n] is not of finite length (i.e., it is not zero outside a finite interval.

System Functions for Systems Characterized by LCCDE

Impulse Response for Rational System Functions

1. ***Infinite Impulse Response (IIR)***

Example 5.6 (A First Order IIR System)

Dr. Shadan Khattak
Department of Electrical Engineering
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System Functions for Systems Characterized by LCCDE

Impulse Response for Rational System Functions

2. ***Finite Impulse Response (FIR)***
    1. $H(z)$ has no poles except at $z = 0$; i.e., $N = 0$.
    2. A PFE is not possible.
    3. $H(z)$ is simply a polynomial in z^{-1} of the form
    $$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$
    4. We assume (without loss of generality) that $a_0 = 1$.
    $$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$
    5. The difference equation is identical to the convolution sum i.e.,
    $$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

System Functions for Systems Characterized by LCCDE

Impulse Response for Rational System Functions

2. ***Finite Impulse Response (IIR)***

Example 5.7 (A Simple FIR System)