EEE 324 Digital Signal Processing

Lectures 13, 14

Frequency Response for Rational System Functions

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- Frequency Response for Rational System Functions

Frequency Response for Rational System Functions

- If a stable LTI system has a rational system function, then its frequency response has the form

$$H\left(e^{j\omega}\right) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

- Substituting $z = e^{j\omega}$ in Eq. (13),

$$H\left(e^{j\omega}\right) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})} \qquad (15)$$

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|(1-c_k e^{-j\omega})\right|}{\prod_{k=1}^{N}\left|(1-d_k e^{-j\omega})\right|} \qquad (15a)$$

- $|H(e^{j\omega})| = \dfrac{(product\ of\ the\ magnitudes\ of\ all\ the\ zero\ factors\ of\ H(z)\ evaluated\ on\ the\ unit\ circle)}{(product\ of\ the\ magnitudes\ of\ all\ the\ pole\ factors\ evaluated\ on\ the\ unit\ circle)}$

Frequency Response for Rational System Functions

- Sometimes, it is convenient to consider magnitude squared of the system function.

$$\left|H\left(e^{j\omega}\right)\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) \qquad (15b)$$

$$\left|H\left(e^{j\omega}\right)\right|^2 = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})(1-c^*{}_k e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})(1-d^*{}_k e^{-j\omega})} \qquad (15c)$$

Frequency Response for Rational System Functions

Log Magnitude Representation of Frequency Response

- The log of Eq. (15a) is:

$$20\log_{10}\left|H\left(e^{j\omega}\right)\right| = 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|\left(1 - c_k e^{-j\omega}\right)\right| - \sum_{k=1}^{N} 20\log_{10}\left|\left(1 - d_k e^{-j\omega}\right)\right| \quad (15d)$$

- The function $20\log_{10}\left|H\left(e^{j\omega}\right)\right|$ is known as the ***log magnitude*** of $H(e^{j\omega})$ and is expressed in decibels (dB).
- It is also known as the ***gain in dB*** i.e.,

$$Gain\ in\ dB = 20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

$$0\ dB \Rightarrow \left|H\left(e^{j\omega}\right)\right| = 1$$

$$\left|H\left(e^{j\omega}\right)\right| = 10^{m} = 20m\ dB$$

$$\left|H\left(e^{j\omega}\right)\right| = 2^{m} \cong 6m\ dB$$

$$\left|H\left(e^{j\omega}\right)\right| < 1 \Rightarrow 20\log_{10}\left|H\left(e^{j\omega}\right)\right| < 0$$

Frequency Response for Rational System Functions

<u>Log Magnitude Representation of Frequency Response</u>

$$Attenuation\ in\ dB = -20log_{10}\left|H\left(e^{j\omega}\right)\right|$$
$$= -Gain\ in\ dB$$

<u>Advantage 1:</u>

- The attenuation is a positive number when the magnitude response is less than unity. e.g., $60\ dB\ attenuation \Rightarrow \left|H\left(e^{j\omega}\right)\right| = 0.001$

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Log Magnitude Representation of Frequency Response

Advantage 2:

- From Eq. (3a),

$$20\log_{10}\left|Y(e^{j\omega})\right| = 20\log_{10}\left|H(e^{j\omega})\right| + 20\log_{10}\left|X(e^{j\omega})\right| \qquad (16)$$

- The frequency response in dB is added to the log magnitude of the i/p FT to find the log magnitude of the o/p FT.
- The effects of both magnitude and phase are additive.

Frequency Response for Rational System Functions

Phase Response for RSF

- The phase response for a rational system function has the form

$$\angle H\left(e^{j\omega}\right) = \angle\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \angle\left[1 - c_k e^{-j\omega}\right] - \sum_{k=1}^{N} \angle\left[1 - d_k e^{-j\omega}\right] \qquad (17)$$

- As in Eq. (15d), the zero factors contribute with a plus sign and the pole factors contribute with a negative sign.

Frequency Response for Rational System Functions

Group Delay for RSF

- The corresponding group delay for a rational system function has the form

$$grd[H(e^{j\omega})] = \sum_{k=1}^{N} \frac{d}{d\omega}\left(arg\left[1 - d_k e^{-j\omega}\right]\right) - \sum_{k=1}^{M} \frac{d}{d\omega}\left(arg\left[1 - c_k e^{-j\omega}\right]\right) \quad (18)$$

- Here arg[] represents the continuous phase.

  Equivalently,

$$grd\left[H\left(e^{j\omega}\right)\right] = \sum_{k=1}^{N} \frac{|d_k|^2 - Re\{d_k e^{-j\omega}\}}{1 + |d_k|^2 - 2Re\{d_k e^{-j\omega}\}} - \sum_{k=1}^{M} \frac{|c_k|^2 - Re\{c_k e^{-j\omega}\}}{1 + |c_k|^2 - 2Re\{c_k e^{-j\omega}\}} \quad (18a)$$

- In Eq. (17), the phase is ambiguous
  - Any integer multiple of 2π can be added to each term at each value of ω without changing the overall value of the complex number.
- The expression for group delay in Eq. (18) involves differentiating the continuous phase.

Frequency Response for Rational System Functions

Phase Response for RSF ***(In terms of the principal value)***

- When the angle of a complex number is calculated using a calculator, the principal value is obtained.
- The principal value of the phase of $H(e^{j\omega})$ is denoted as $ARG[H(e^{j\omega})]$, where

$$-\pi < ARG\left[H\left(e^{j\omega}\right)\right] \leq \pi$$

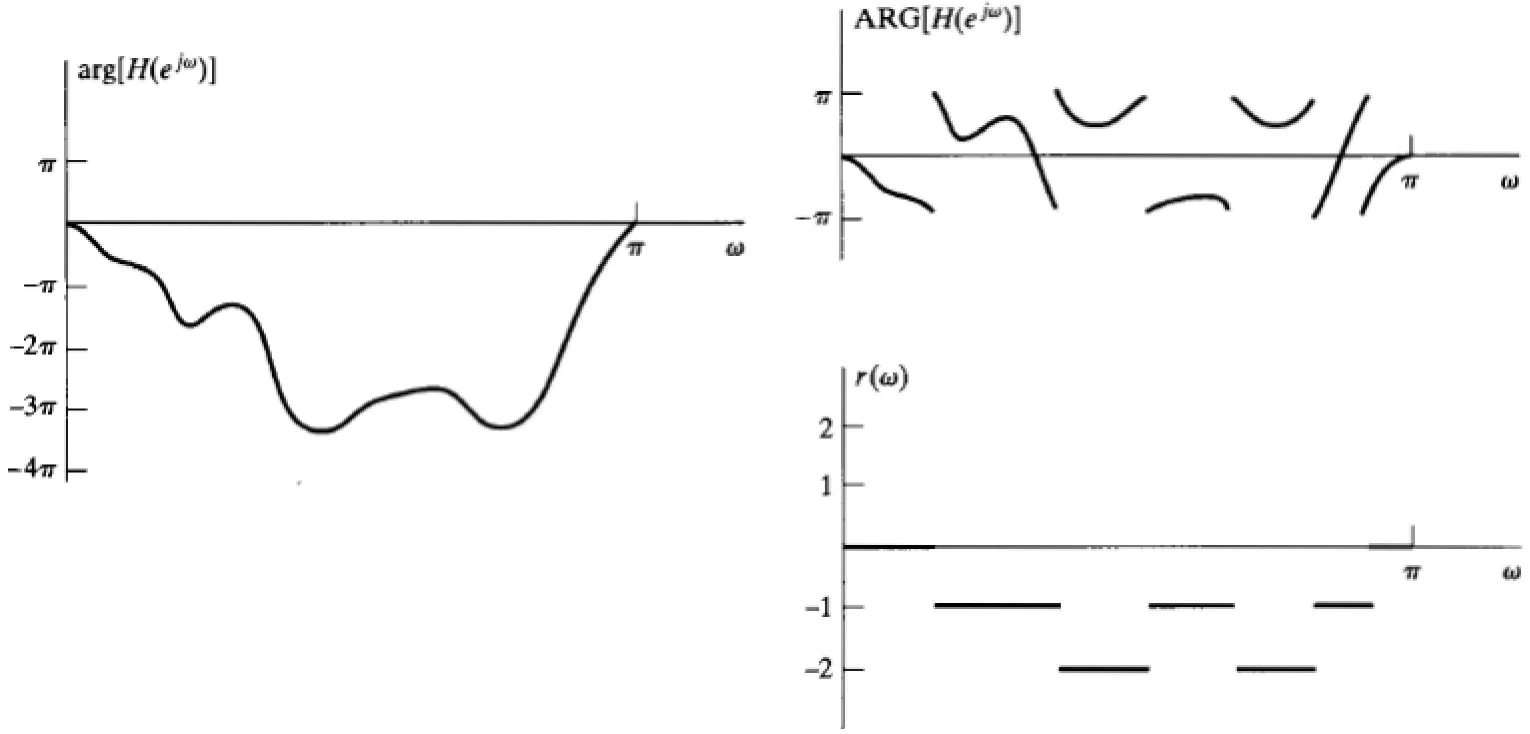
- The correct complex value of the function $H(e^{j\omega})$ in terms of the principal value is:

$$angle\left(H\left(e^{j\omega}\right)\right) = ARG\left[H\left(e^{j\omega}\right)\right] + 2\pi r(\omega)$$

Where $r(\omega)$ can be a positive or a negative integer which can be different at each value of ω.

Frequency Response for Rational System Functions

Phase Response for RSF ***(In terms of the principal value)***



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Frequency Response for Rational System Functions

Phase Response for RSF ***(In terms of the principal value)***

$$ARG\left[H\left(e^{j\omega}\right)\right] = ARG\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} ARG\left[1 - c_k e^{-j\omega}\right] - \sum_{k=1}^{N} ARG\left[1 - d_k e^{-j\omega}\right] + 2\pi r(\omega) \qquad (19)$$

- In general, the principal value of a sum of angles is not equal to the sum of the principal values of the individual angles.

Also,

$$ARG\left[H\left(e^{j\omega}\right)\right] = \arctan\left[\frac{H_I\left(e^{j\omega}\right)}{H_R\left(e^{j\omega}\right)}\right]$$

$$H_R\left(e^{j\omega}\right): \quad Real\ Part\ of\ H\left(e^{j\omega}\right)$$

$$H_I\left(e^{j\omega}\right): Imaginary\ Part\ of\ H\left(e^{j\omega}\right)$$

Frequency Response for Rational System Functions

<u>Group Delay for RSF</u> (In terms of the principal value)

$$grd[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

- Except at discontinuities of $ARG[H(e^{j\omega})]$,

$$\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \frac{d}{d\omega}\{\mathrm{ARG}[H(e^{j\omega})]\}$$

- Hence, group delay can be obtained from the principle value by differentiating, except at discontinuities.

Practice Problems

Problems 5.1 – 5.15, 5.17, 5.19, 5.35, 5.36 (Oppenheim)