EEE 324 Digital Signal Processing

Lecture 15

Frequency Response of a Single Pole or Zero

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Frequency Response for Rational System Functions

1. Frequency Response of a Single Zero or Pole

- Eq. (15d), Eq. (17), and Eq. (18d) represent the magnitude in dB, the phase, and the group delay, respectively, as a sum of contributions from each of the poles and zeros of the system.

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Frequency Response for Rational System Functions

1. Frequency Response of a Single Zero or Pole

- Let's first examine the properties of a single factor of the form $(1 - re^{j\theta}e^{-j\omega})$ where r is the radius and θ is the angle of the pole or zero in the z-plane.
- This is typical of either a pole or zero at a radius r and angle θ in the z-plane.
- The square magnitude of such a factor,

$$\left|1 - re^{j\theta}e^{-j\omega}\right|^2 = (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega})$$

$$= 1 + r^2 - 2rcos(\omega - \theta) \qquad (19)$$

Frequency Response for Rational System Functions

1. Frequency Response of a Single Zero or Pole

- Since, for any complex quantity C,

$$10\log_{10}|C|^2 = 20\log_{10}|C|$$

The log magnitude in dB is

$$20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right| = 10\log_{10}[1 + r^2 - 2r\cos(\omega - \theta)] \qquad (20)$$

The principal value phase is

$$ARG\left[1 - re^{j\theta}e^{-j\omega}\right] = \arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] \qquad (21)$$

The group delay is

$$grd\left[1 - re^{j\theta}e^{-j\omega}\right] = \frac{r^2-r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)} = \frac{r^2-r\cos(\omega-\theta)}{\left|1-re^{j\theta}e^{-j\omega}\right|^2} \qquad (22)$$

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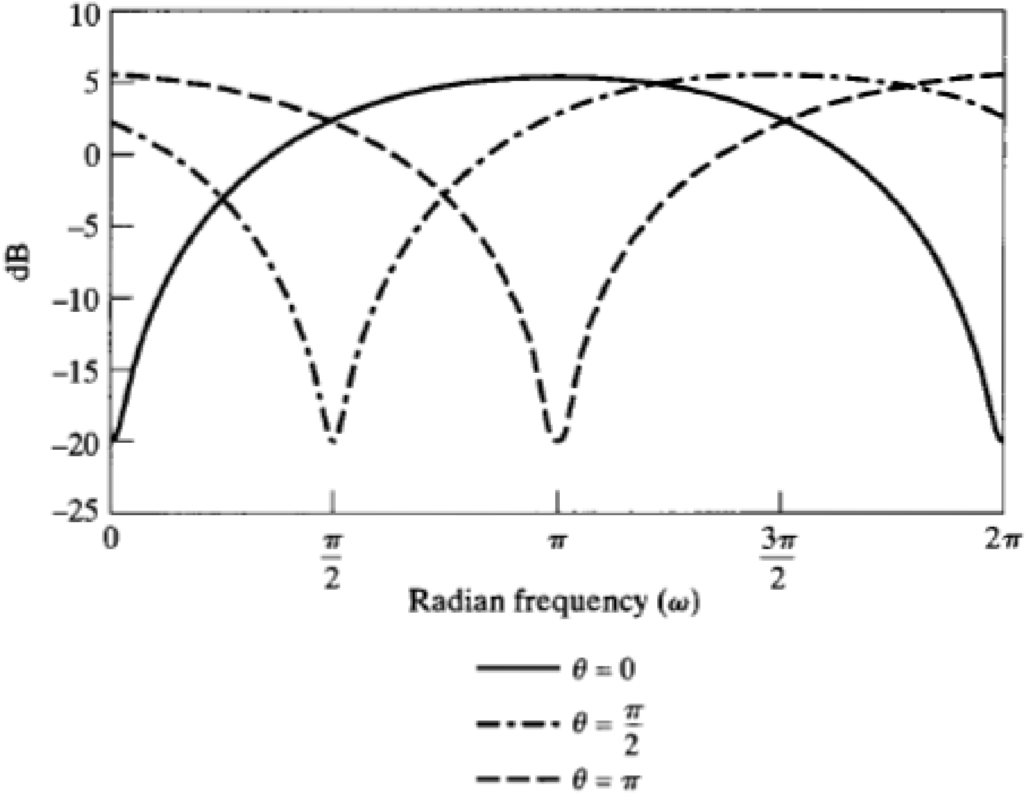
Frequency Response for Rational System Functions

1. ***Frequency Response of a Single Zero or Pole***

- **Magnitude as a function of ω for different values of θ**
- When r is fixed (here, $r = 0.9$), the function of Eq. (20)
  - dips sharply in the vicinity of $\omega = \theta$.
    - The minimum value is $20\log_{10}|1 - r|$
    - For r = 0.9, min = $-20\ dB$
  - Has a maximum value when $(\omega - \theta) = \pi$
    - The maximum value is $20\log_{10}(1 + r)$
    - For $r = 0.9$, max = 5.57 dB



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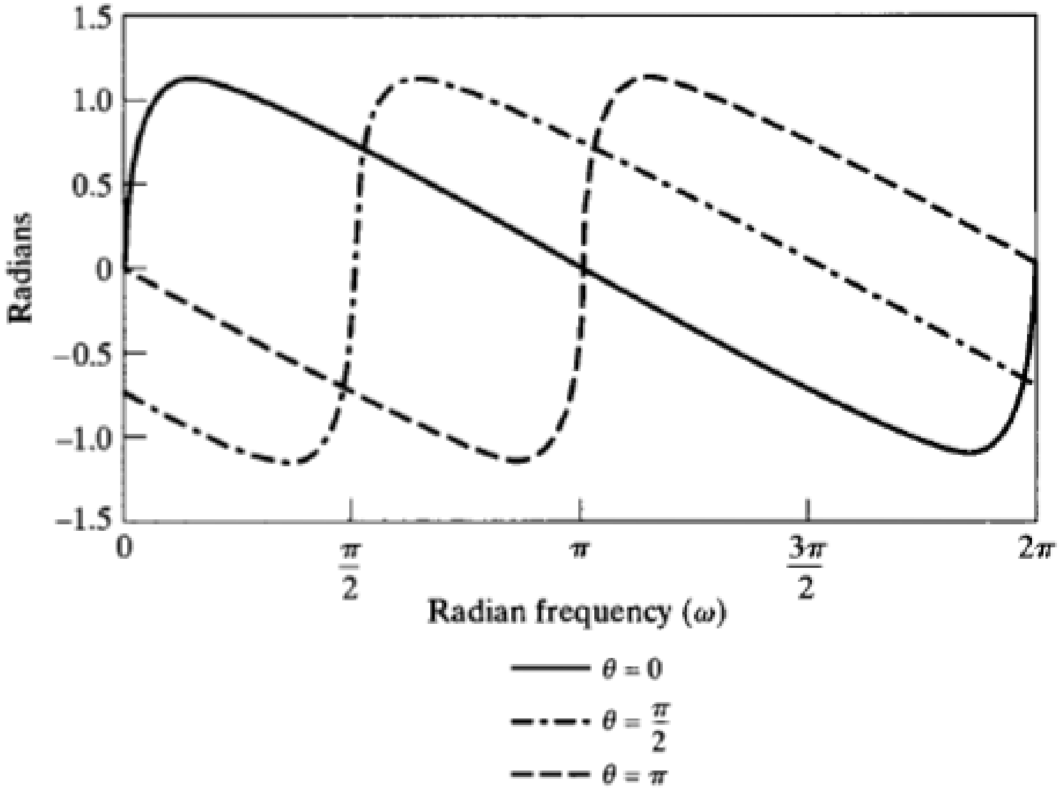
Frequency Response for Rational System Functions

1. ***Frequency Response of a Single Zero or Pole***

- <u>**Phase as a function of ω**</u>
- Phase is 0 at $\omega = \theta$.
- For fixed r, the function shifts with θ.



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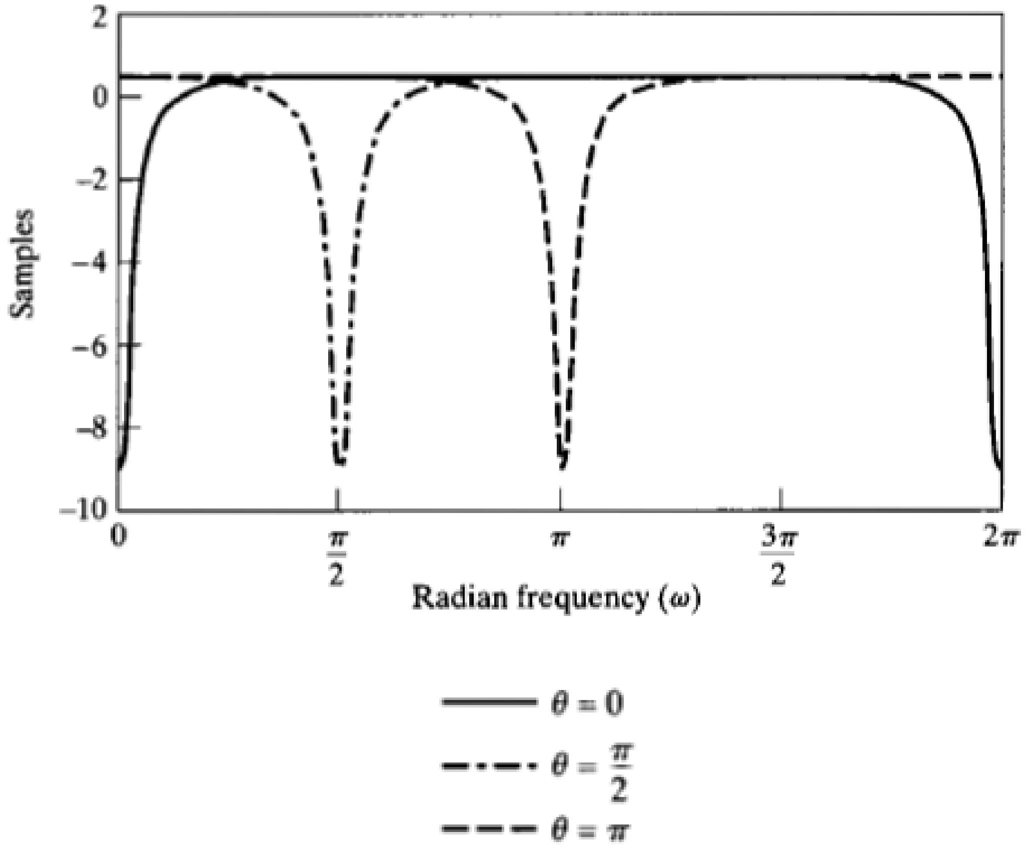
Frequency Response for Rational System Functions

1. ***Frequency Response of a Single Zero or Pole***

- <u>**Group Delay as a function of ω**</u>
- High positive slope of the phase around $\omega = \theta$ corresponds to a large negative peak in the group delay function at $\omega = \theta$



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Frequency Response for Rational System Functions

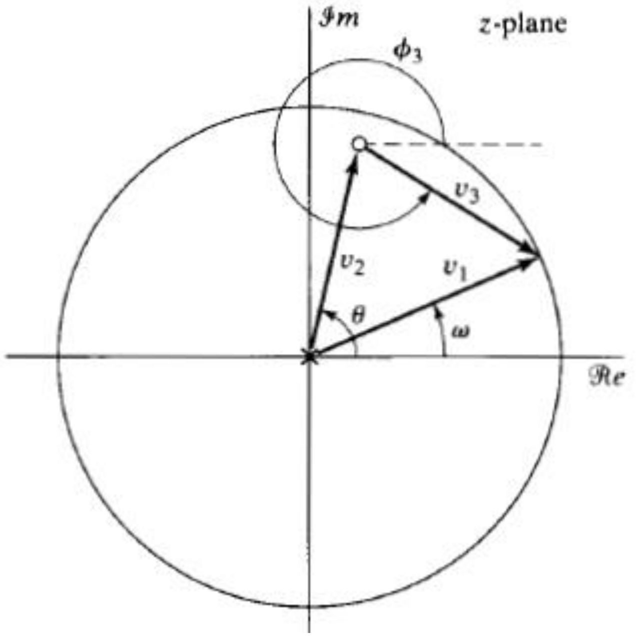
2. *The Contribution of a Single Zero Factor (Geometric Construction)*

Consider a first order system function of the form

$$H(z) = \left(1 - re^{j\theta}z^{-1}\right) = \frac{z - re^{j\theta}}{z}$$

$$r < 1$$

- Pole: $z = 0$
- Zero: $z = e^{j\theta}$
- v_1 represents $e^{j\omega}$
  - It is a ***pole vector*** since it connects a pole with the unit circle.
- v_2 represents $re^{j\theta}$
- $v_3 = v_1 - v_2$ represents $(e^{j\omega} - re^{j\theta})$
  - Here, v_3 is a ***zero vector*** since it connects a zero with the unit circle



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Frequency Response for Rational System Functions

2. *The Contribution of a Single Zero Factor (Geometric Construction)*

- ***Magnitude:***
  - The magnitude of the complex number $\frac{e^{j\omega}-re^{j\theta}}{e^{j\omega}}$ is the ratio of the magnitudes of the vectors v_3 and v_1 i.e.,

$$\left|1-re^{j\theta}e^{-j\omega}\right| = \left|\frac{e^{j\omega}-re^{j\theta}}{e^{j\omega}}\right| = \frac{|v_3|}{|v_1|}$$

Since $|v_1| = 1$,

$$\left|1-re^{j\theta}e^{-j\omega}\right| = \left|\frac{e^{j\omega}-re^{j\theta}}{e^{j\omega}}\right| = |v_3|$$

- *The contribution of a single zero factor $\left(1-re^{j\theta}z^{-1}\right)$ to the magnitude function at frequency ω is the length of the zero vector v_3 from the zero to the point $z=e^{j\omega}$ on the unit circle.*
  - *The vector has a minimum length when $\omega=\theta$.*
  - *As the pole, in this case, lies at z=0, it does not have any effect on the magnitude response.*

Frequency Response for Rational System Functions

2. *The Contribution of a Single Zero Factor (Geometric Construction)*

- ***Phase:***

$$angle\left(1 - re^{j\theta}e^{-j\omega}\right) = angle\left(\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}\right)$$

$$= angle\left(e^{j\omega} - re^{j\theta}\right) - angle\left(e^{j\omega}\right)$$

$$= \emptyset_3 - \omega$$ (See the figure on Slide 9)

- Phase function is equal to the difference between the angle of the zero vector from the zero at $re^{j\theta}$ to the point $z = e^{j\omega}$ and the angle of the pole vector from the pole at $z = 0$ to the point $z = e^{j\omega}$

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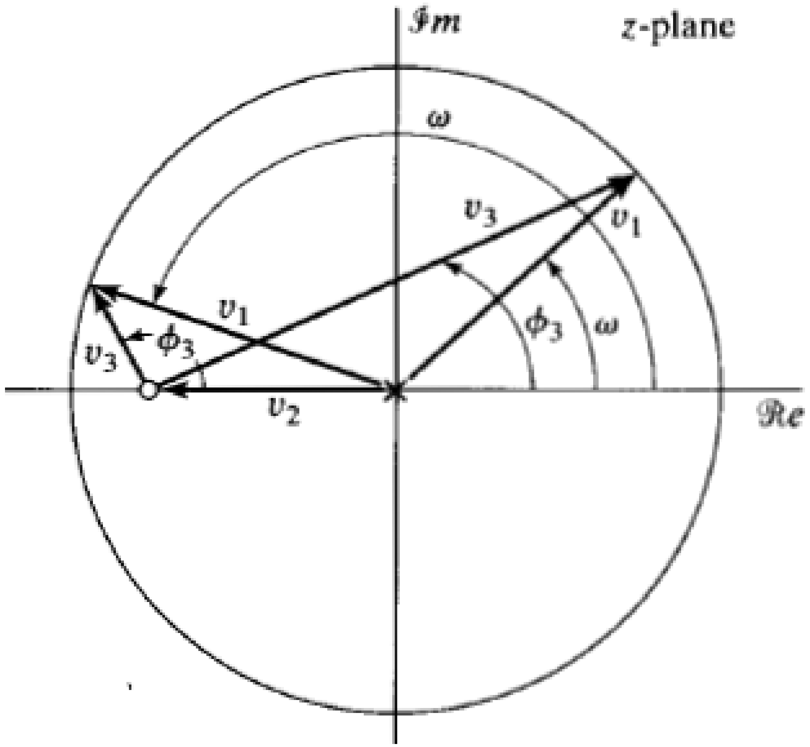
Frequency Response for Rational System Functions

2. ***The Contribution of a Single Zero Factor (Geometric Construction)***

Example ($\theta = \pi$)

- Two different values of ω have been considered.
    - As ω increases from 0, the magnitude of the vector v_3 decreases until it reaches a minimum at $\omega = \pi$.
    - The angle of v_3 increases more slowly than ω at first, so that the phase curve starts out negative, then when ω is close to π, the angle of v_3 increases more rapidly than ω, thereby accounting for the steep positive slope of the phase function around $\omega = \pi$



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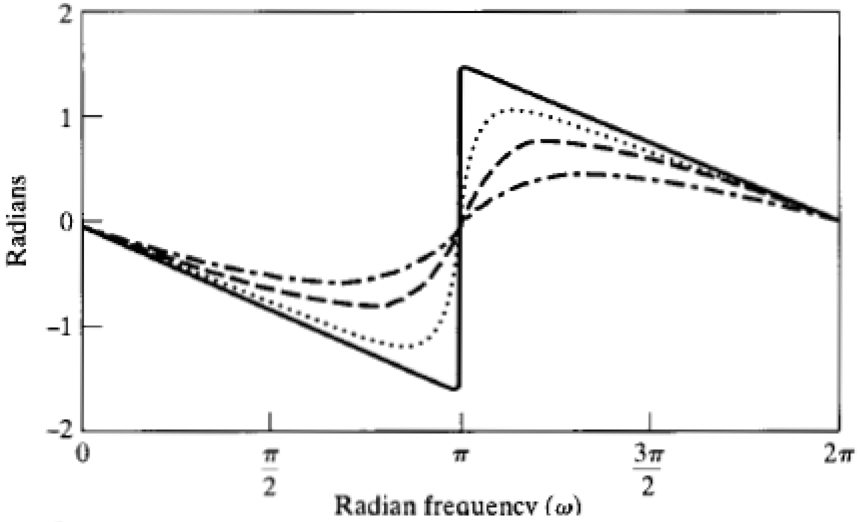
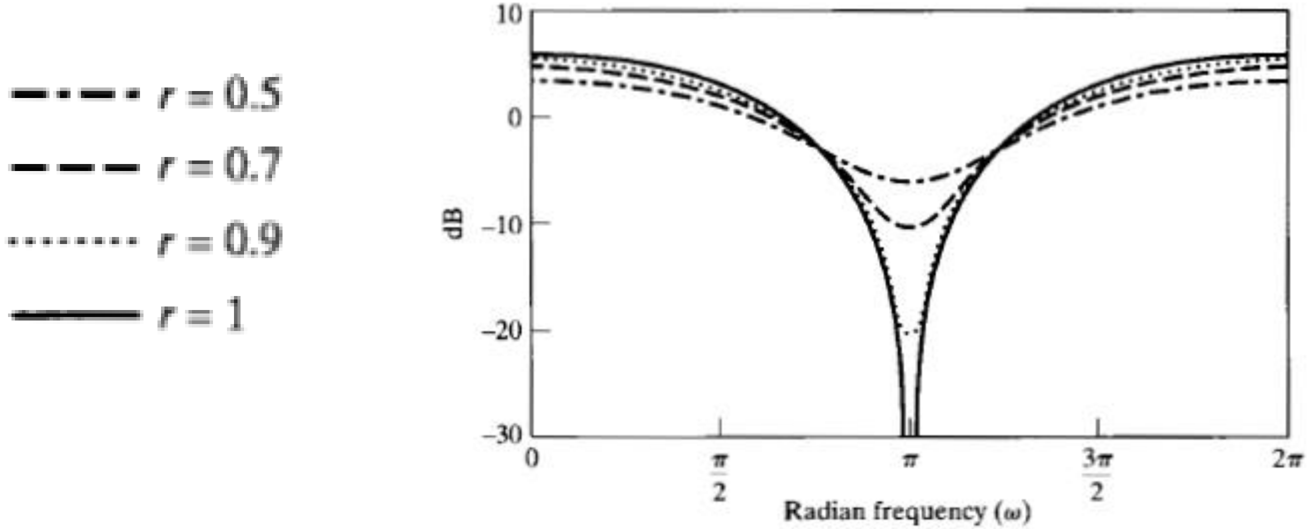
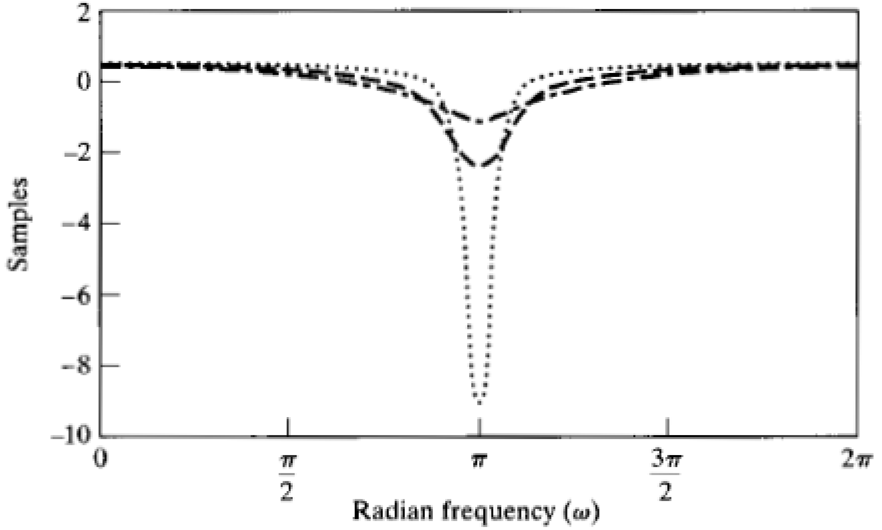
2. ***The Contribution of a Single Zero Factor (Geometric Construction)***

Example $(r = chaning, \theta = \pi\ (fixed))$







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Frequency Response for Rational System Functions

2. ***The Contribution of a Single Zero Factor (Geometric Construction)***

(Dependence on the radius r *for the case* $\theta = \pi$)

- The log magnitude function dips more sharply as r becomes closer to 1.
  - Approaches $-\infty$ at $\omega = \pi$ as r approaches 1.
- The phase function has positive slope around $\omega = \theta$ which becomes infinite as r approaches 1.
  - Thus for $r = 1$, the phase function is discontinuous with a jump of π radians at $\omega = \theta$
  - Away from $\omega = \theta$, the slope of the phase function is negative.
- Since the GD is the negative of the slope of the phase curve, the GD is negative around $\omega = \theta$
  - it dips sharply as r approaches 1.
  - As we move away from $\omega = \theta$, the GD becomes positive and relatively flat.
  - When $r = 1$, $GD = \frac{1}{2}$ everywhere except at $\omega = \theta$ where it is undefined.

Frequency Response for Rational System Functions

2. ***The Contribution of a Single Zero Factor (Geometric Construction)***

- A single zero at $z = -1$.
- Two different frequencies:
  - $\omega = (\pi - \varepsilon)$
  - $\omega = (\pi + \varepsilon)$

  Where ε is a very small number.

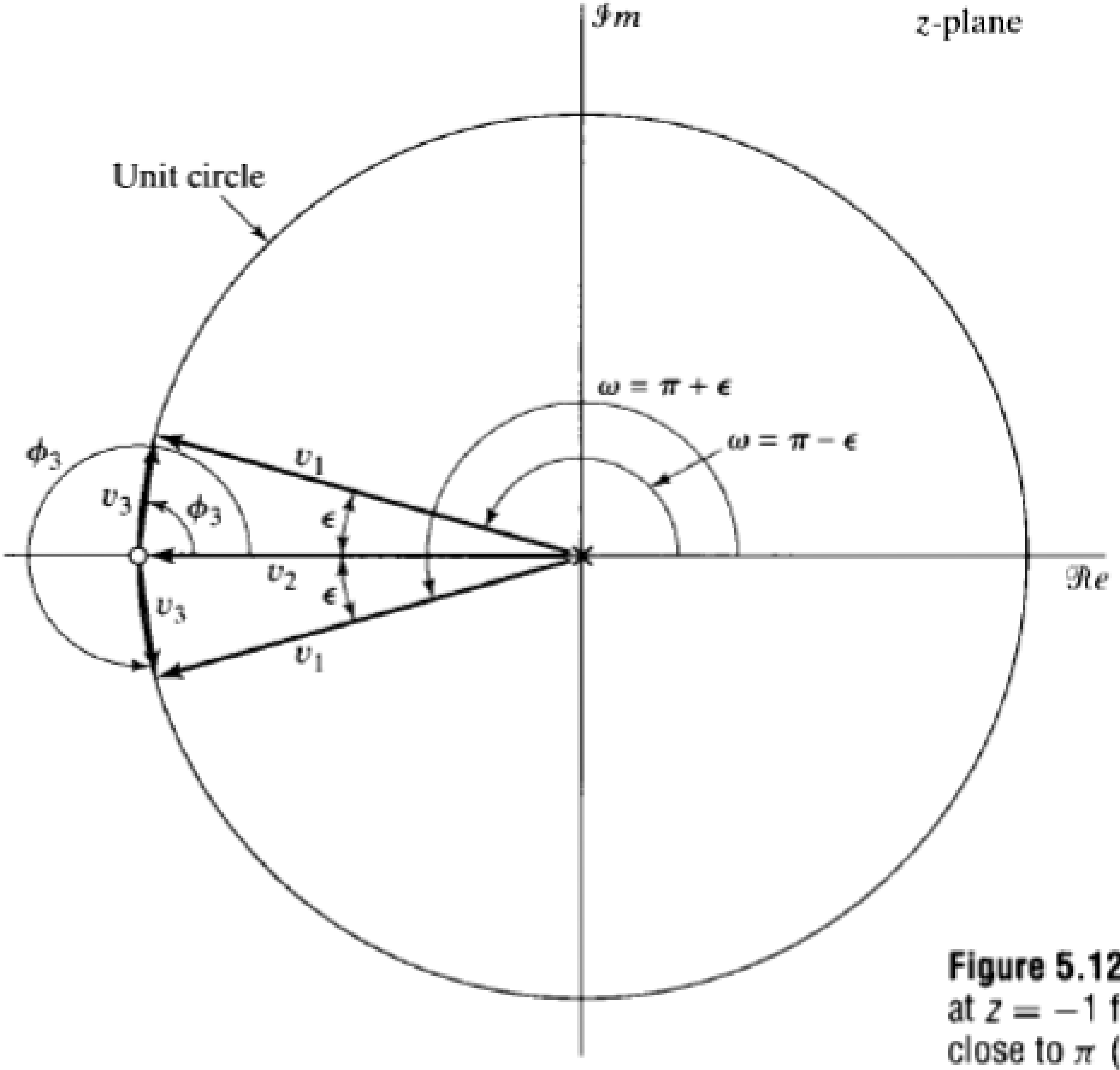


Figure 5.12 z-plane vectors for a zero at $z = -1$ for two different frequencies close to π ($\omega = \pi - \varepsilon$ and $\pi + \varepsilon$).

Frequency Response for Rational System Functions

The Contribution of a Single Zero Factor (Geometric Construction)

- Two observations:
  - Length of the vector v_3 approaches 0 as ω approaches the angle of the zero vector $(\varepsilon \to 0)$
    - Hence, the multiplicative contribution to the FR is 0 $(-\infty\ dB)$
  - The vector v_3 changes its angle discontinuously by π radians as ω goes from $(\pi - \varepsilon)$ to $(\pi + \varepsilon)$

Frequency Response for Rational System Functions

The Contribution of a Single Zero Factor (The case of $r > 1$)

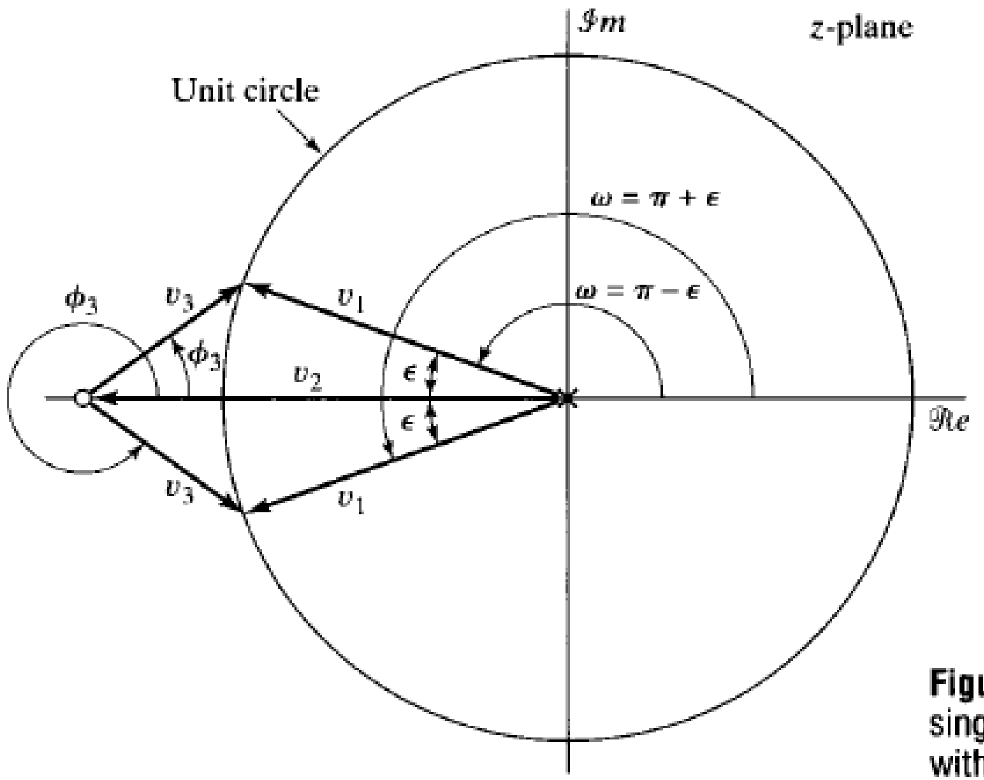
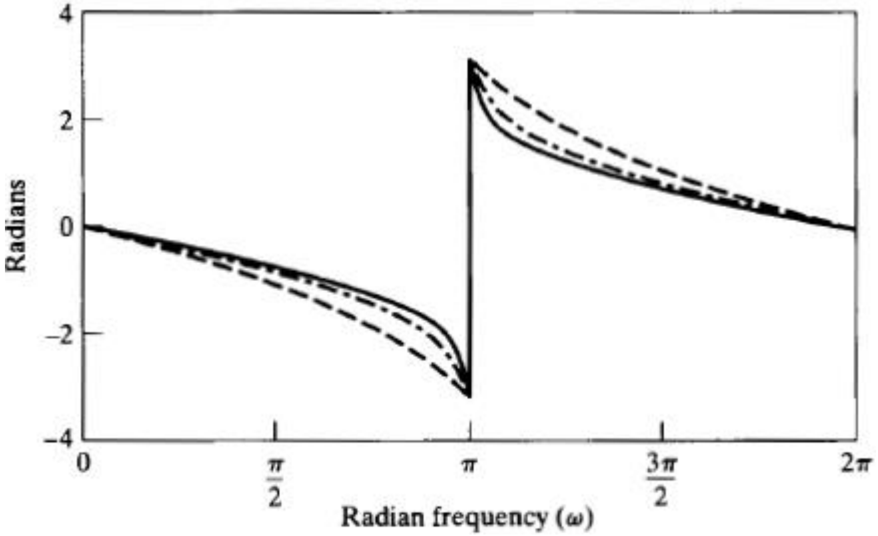


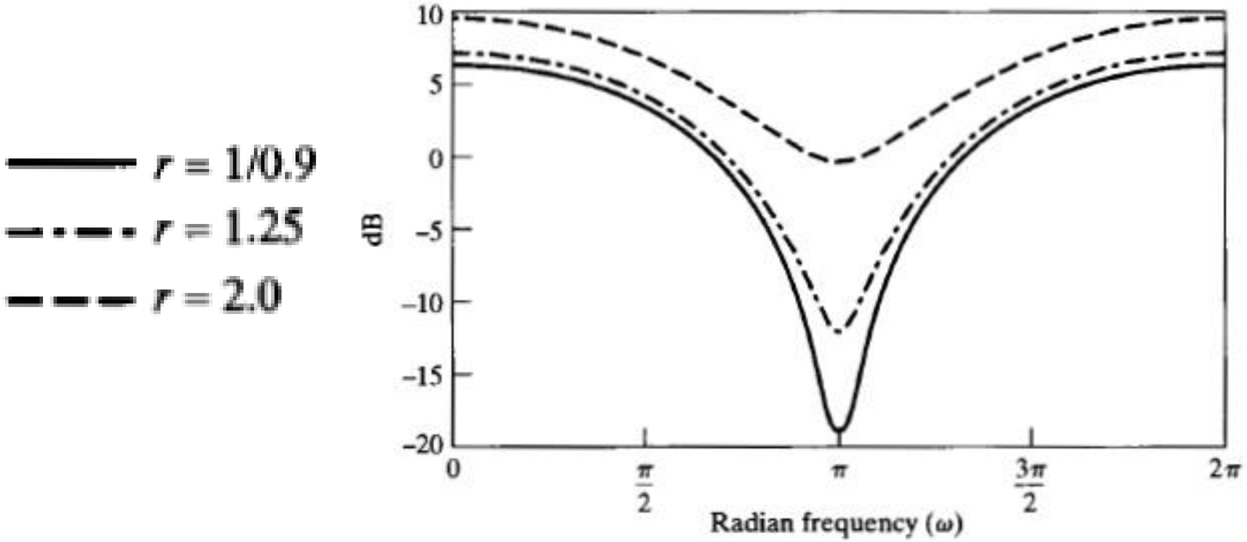
Figure 5.14 z-plane vectors for a single zero evaluated on the unit circle, with $\theta = \pi$, $r > 1$.

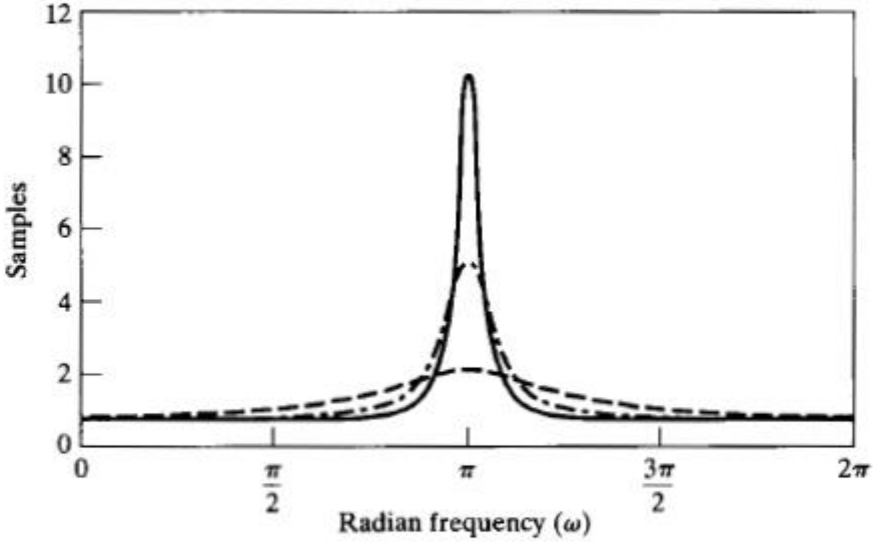
Frequency Response for Rational System Functions

The Contribution of a Single Zero Factor

The case of $r > 1$







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The Contribution of a Single Zero Factor

The case of $r > 1$

- The log magnitude function behaves similarly to the case $r < 1$.
  - i.e., it dips more sharply as $r \to 1$.
- The phase function shows a discontinuity of 2π radians at $\omega = \theta$ for all values of $r > 1$.
- All the phase curves have negative slopes.
  - Hence, the GD is positive for all values of ω

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The Contribution of a Single Zero Factor

The case of a pole

- In the previous discussion, we have only considered a single zero factor.
- If the factor represents a pole, all the contributions will enter with an opposite sign.
- Hence, the contribution of a pole would be the negative of the curves in the preceding figures.
- The dependence on r would be the same as for a zero i.e.,
  - The closer r is to 1, the more peaked will be the contribution to the magnitude function.
- For stable and causal systems, there will be no poles outside the unit circle i.e.,
  - r will always be less than 1.

Frequency Response for Rational System Functions

Examples with Multiple Poles or Zeros

Example 8 (Second Order IIR System):

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Examples with Multiple Poles or Zeros

Example 9 (Second Order FIR System):

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Examples with Multiple Poles or Zeros

Example 10 (Third Order IIR System):

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