EEE 324 Digital Signal Processing

Lectures 16, 17

Relationship Between Magnitude and Phase
All Pass Systems

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Relationship Between Magnitude and Phase

- Generally, knowledge about the magnitude provides no information about the phase, and vice versa.
- But, for systems described by LCCDE (i.e., Rational System Functions), there exists some constraints between magnitude and phase.

1. If MR and the number of poles and zeros are known, then there are only a finite number of choices for the PR. Also,
2. If PR and the number of poles and zeros are known, then, to within a scale factor, there are only a finite number of choices for the MR.
3. ***Under the minimum phase constraint***, the MR specifies the phase uniquely and the PR specifies the magnitude to within a scale factor.

Relationship Between Magnitude and Phase

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega})$$
$$= H(z)H^*\left(\frac{1}{z^*}\right)\bigg|_{z=e^{j\omega}}$$

For a rational system,

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

$$H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-{c_k}^* z)}{\prod_{k=1}^{N}(1-{d_k}^* z)}$$

Hence, square of the magnitude of the FR is evaluation on the unit circle of the z-transform

$$C(z) = H(z)H^*\left(\frac{1}{z^*}\right)$$

$$C(z) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k z^{-1})(1-{c_k}^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-{d_k}^* z)}$$

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Relationship Between Magnitude and Phase

- If we are given $\left|H(e^{j\omega})\right|^2$, then by replacing $e^{j\omega}$ by z, we can construct $C(z)$.
- What information can $C(z)$ give us about $H(z)$?
  - For each pole d_k of $H(z)$, there is a pole of $C(z)$ at d_k and $(d_k{}^*)^{-1}$.
  - Similarly, for each zero c_k , there is a zero of $C(z)$ at c_k and $(c_k{}^*)^{-1}$.
  - So, the poles and zeros of $C(z)$ occur in conjugate reciprocal pairs with one element of each pair associated with $H(z)$ and one element of each pair associated with $H^*\left(\frac{1}{z^*}\right)$.
  - If one element of each pair is inside the unit circle, then the other will be outside the unit circle.
  - Another alternative is that both the elements lie on the unit circle in the same location.

Relationship Between Magnitude and Phase

- If $H(z)$ is assumed to be causal and stable,
  - All poles must lie inside the unit circle.
  - Poles of $H(z)$ can be identified from the poles of $C(z)$.
  - With this constraint alone, Zeros of $H(z)$ cannot be uniquely identified from zeros of $C(z)$.

Relationship Between Magnitude and Phase

Example 5.11 (Systems with the same $C(z)$)

Same magnitude squared response

Different phase responses

Relationship Between Magnitude and Phase

Example 5.12

All Pass Systems

- A stable system function of the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \qquad (1)$$

has a MR that is independent of ω. This can be seen by writing $H_{ap}(e^{j\omega})$ in the form

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- The term $e^{-j\omega}$ has unity magnitude.
- For the term $\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$, the numerator and denominator are complex conjugates of each other and hence have the same magnitude. So

$$|H_{ap}(e^{j\omega})| = 1$$

All Pass Systems

- A system for which the MR is constant is called an ***all pass system (APS)***.
  - Because the system passes all of the frequency components of its input with constant gain or attenuation.
- The most general form of the system function of an APS with a real valued IR is a product of factors like Eq. (1), with complex poles being paired with their conjugates.

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - {e_k}^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - {e_k}^* z^{-1})}$$

A is a positive constant

$d_k s$ are the real poles

$e_k s$ are the complex poles

All Pass Systems

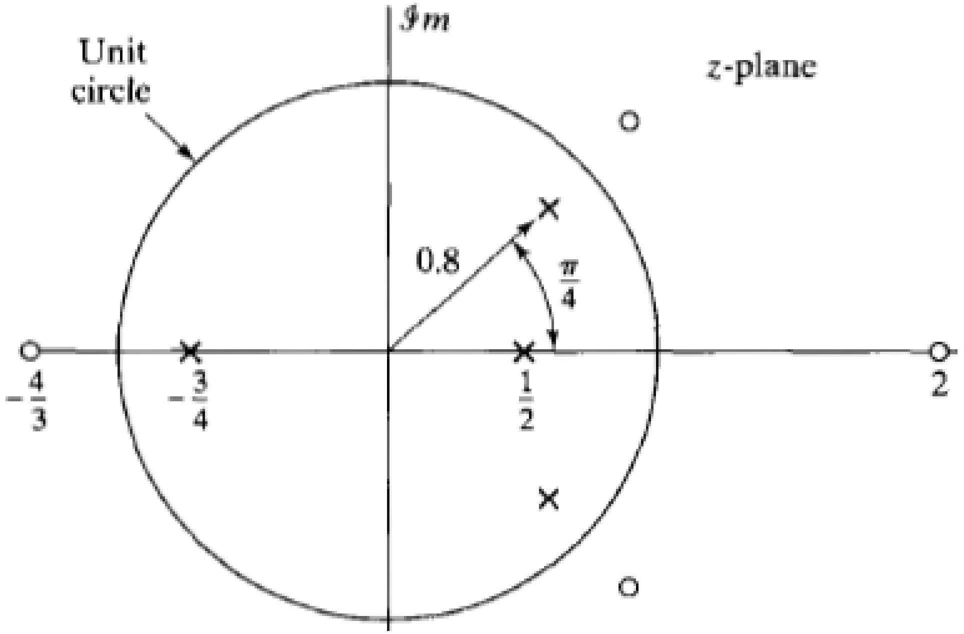
- For causal and stable APS,

$$|d_k| < 1 \text{ and } |e_k| < 1$$

- APS have $M = N = 2M_c + M_r$
- In this figure,

$$M_r = 2 \text{ and } M_c = 1$$

- Note that each pole is paired with a conjugate reciprocal zero.



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All Pass Systems

- The FR of a general APS can be expressed in terms of the FR of first order APSs.
- For a causal APS, each of the first order terms consist of a pole inside the unit circle and a zero at the conjugate reciprocal location.
- The MR of such a term is 1.
  - And hence, the log magnitude in dB is 0.

All Pass Systems

- The Phase function of a first order APS is

$$angle\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = -\omega - 2\arctan\left[\frac{rsin(\omega-\theta)}{1 - rcos(\omega-\theta)}\right]$$

- The Phase function of a second order APS with poles at $z = re^{j\theta}$ and $z = re^{-j\theta}$ is

$$angle\left[\frac{(e^{-j\omega} - re^{-j\theta})(e^{-j\omega} - re^{j\theta})}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}\right] = -2\omega - 2\arctan\left[\frac{rsin(\omega-\theta)}{1 - rcos(\omega-\theta)}\right] - 2\arctan\left[\frac{rsin(\omega+\theta)}{1 - rcos(\omega+\theta)}\right]$$

All Pass Systems

<u>Example 5.13 (First and Second Order APS</u>

<u>*First Order*</u>

Note: when r>1, the zero causes a negative slope around $\omega = \theta$ *and when r<1, the pole causes a negative slope around* $\omega = \theta$

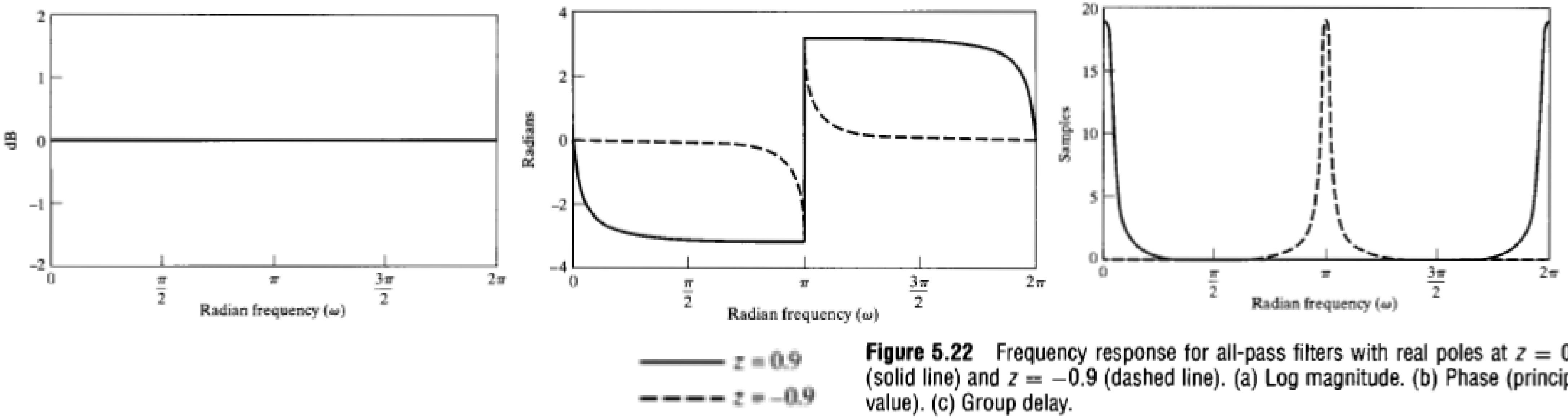

Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

All Pass Systems

<u>Example 5.13 (First and Second Order APS</u>

<u>*(Second Order)*</u>

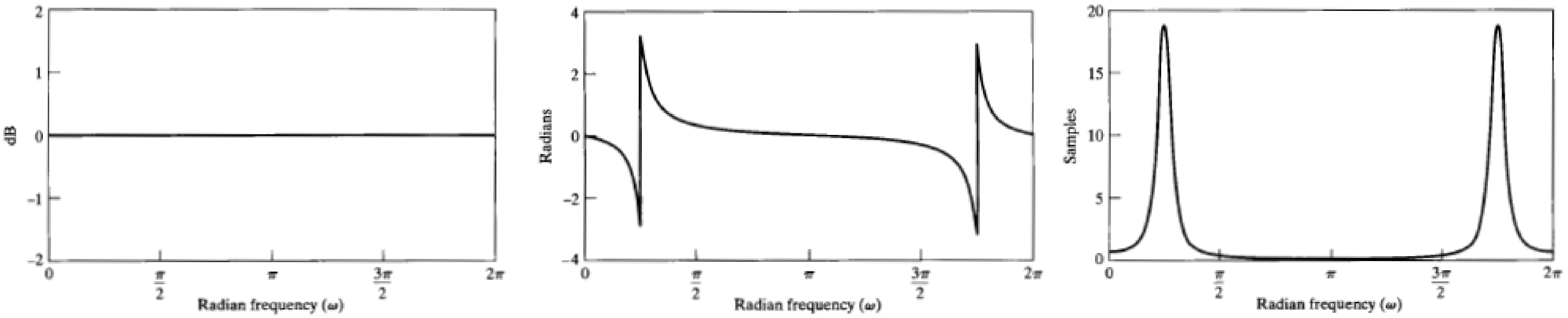

Figure 5.23 Frequency response of second-order all-pass system with poles at $z = 0.9e^{\pm j\pi/4}$. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

All Pass Systems

<u>Example 5.13 (First and Second Order APS</u>

<u>*(Third Order)*</u>

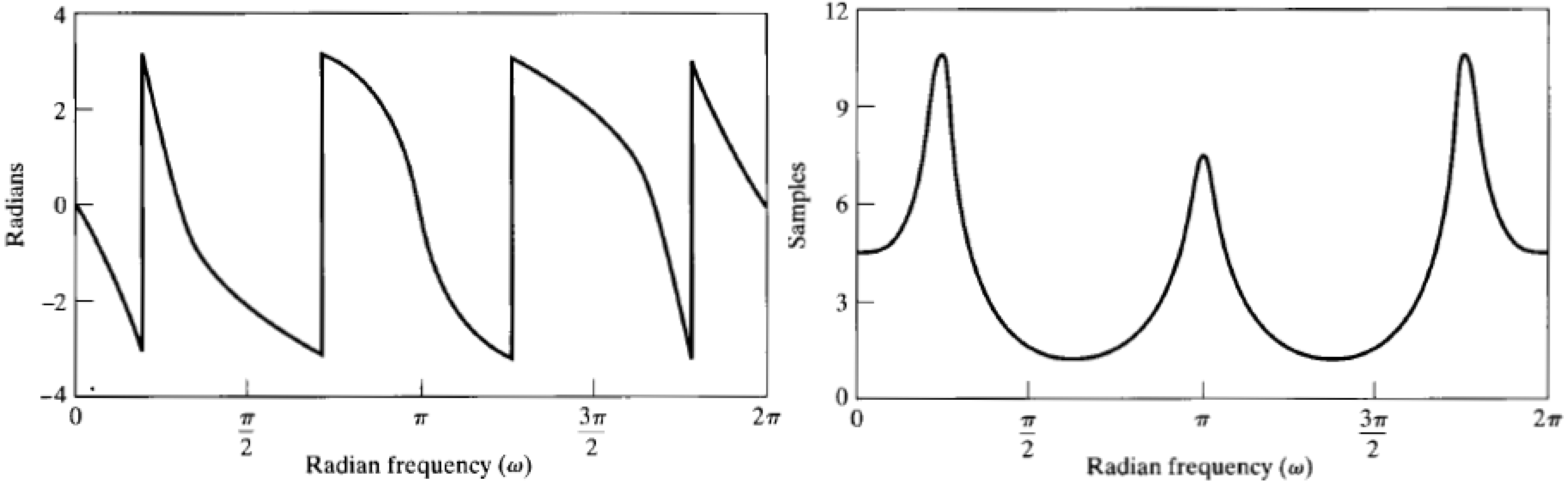

Figure 5.24 Frequency response for an all-pass system with the pole–zero plot in Figure 5.21. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

(Log Mag. not shown)

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All Pass Systems

Example 5.13 (First and Second Order APS

- ***General Properties of all causal APS:***

1. *Continuous phase of a causal APS is always non-positive for* $0 < \omega < \pi$ *. For example,*
   a) *Phase is non-positive for* $0 < \omega < \pi$ *(See Fig. 5.22(b))*
   b) *In Fig. 5.23(b), if the discontinuity of* 2π *resulting from the computation of the principal value is removed, the resulting continuous phase curve is non-positive for* $0 < \omega < \pi$

All Pass Systems

<u>**Example 5.13 (First and Second Order APS**</u>

- ***General Properties of all causal APS:***

2. *For a causal and stable APS with $r < 1$, the GD contributed by a single causal APS factor is always positive.*
3. *Since the GD of a higher order APS will be a sum of positive terms, it is generally true that GD of a causal rational APS is always positive (E.g., Fig. 5.22(c), Fig. 5.23(c), Fig. 5.24(c).*
   - *The positivity of the GD of a causal APS is the basis for a simple proof of the negativity of the phase of such a system.*
   - *The positivity of the GD and the non-positivity of the continuous phase are important properties of APS.*

All Pass Systems

Uses of APS

1. They can be used as compensators for phase (or GD) distortion.
2. They are useful in the theory of minimum phase systems (MPS).
3. They are useful in transforming frequency selective LPF into other frequency selective forms.
4. They are useful in obtaining variable cut-off frequency selective filters.
5. These applications are discussed in detail in Chapter 7.