EEE 324 Digital Signal Processing

Lecture 18

Minimum Phase Systems

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Contents

- Minimum Phase Systems

Minimum Phase Systems

- For certain classes of problems, it is useful to impose the restriction that the inverse system also be stable and causal.
  - For such systems, both poles and zeros must be inside the unit circle.
  - Such systems are referred to as ***Minimum Phase Systems***.
- If we are given a magnitude squared function and we know that the system is an MPS, then $H(z)$ is uniquely determined and will consist of all the poles and zeros of $C(z)$ that lie inside the unit circle.

Minimum Phase Systems

Minimum Phase and All Pass Decomposition

- Any rational system function can be expressed as

$$H(z) = H_{min}(z)H_{ap}(z)$$

$H_{min}(z)$: Minimum Phase System

$H_{ap}(z)$: All Pass System

Minimum Phase Systems

Minimum Phase and All Pass Decomposition

To show this, suppose that $H(z)$ has one zero outside the unit circle at $z = 1/c^*$, where $|c| < 1$, and the remaining poles and zeros are inside the unit circle. So

$$H(z) = H_1(z)(z^{-1} - c^*)$$

Where $H_1(z)$ is an MPS. Equivalently,

$$H(z) = \frac{H_1(z)(1 - cz^{-1})(z^{-1} - c^*)}{(1 - cz^{-1})}$$

Since $|c| < 1, H_1(z)(1 - cz^{-1})$ is also an MPS.

On the other hand, the term $\frac{(z^{-1} - c^*)}{(1 - cz^{-1})}$ is an APS.

Minimum Phase Systems

Minimum Phase and All Pass Decomposition

- Hence, for stable, causal systems, in case many zeros are outside the unit circle, any system function can be expressed as:

$$H(z) = H_{min}(z)H_{ap}(z) \qquad (1)$$

Where

- $H_{min}(z)$ contains the poles and zeros of $H(z)$ that lie inside the unit circle, plus zeros that are the conjugate reciprocals of the zeros of $H(z)$ that lie outside the unit circle.
- $H_{ap}(z)$ is comprised of all the zeros of $H(z)$ that lie outside the unit circle, together with poles to cancel the reflected conjugate reciprocal zeros in $H_{min}(z)$

Minimum Phase Systems

Minimum Phase and All Pass Decomposition

Example 5.14

$$H_1(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

$$H_2(z) = \frac{(1 + \frac{3}{2}e^{\frac{j\pi}{4}}z^{-1})(1 + \frac{3}{2}e^{\frac{-j\pi}{4}}z^{-1})}{1 - \frac{1}{3}z^{-1}}$$

Minimum Phase Systems

Frequency Response Compensation

- If a signal has been distorted by an LTI system with an undesirable FR, then, generally, a compensating system is used to process the distorted signal.
- E.g., in transmitting signals over a communication channel.

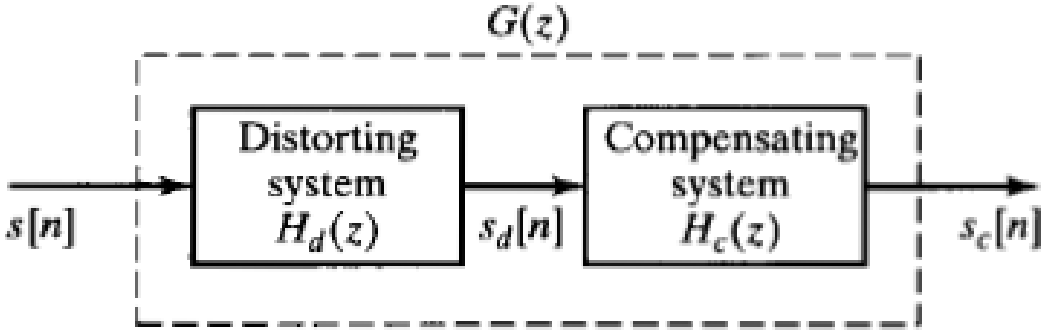

Figure 5.25 Illustration of distortion compensation by linear filtering.

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Minimum Phase Systems

<u>Frequency Response Compensation</u>

- If perfect compensation is achieved, then $s_c[n] = s[n]\ i.e., H_c(z)$ is the inverse of $H_d(z)$.
- If we assume that the distorting system is stable and causal and require the compensating system to be stable and causal, then perfect compensation is possible only if $H_d(z)$ is a minimum phase system, so that it has a stable, causal inverse.

Minimum Phase Systems

Frequency Response Compensation

- How do we construct an MPS?
  - If we assume that $H_d(z)$ is known or it is approximated as an RSF, we can form an MPS $H_{d\,min}(z)$ by reflecting all the zeros of $H_d(z)$ that are outside the unit circle to their conjugate reciprocal locations inside the unit circle.
  - $H_d(z)$ and $H_{d\,min}(z)$ have the same MR and are related through an APS $H_{ap}(z)$ i.e.,

$$H_d(z) = H_{d\,min}(z)H_{ap}(z) \qquad (2)$$

Then the compensating filter will be

$$H_c(z) = \frac{1}{H_{d\,min}(z)}$$

And the overall system function is

$$G(z) = H_d(z)H_c(z) = H_{ap}(z) \qquad (3)$$

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Minimum Phase Systems

Frequency Response Compensation

- So $G(z)$ is an APS.
- Consequently,
  - The MR is exactly compensated for, while the PR is modified to $angle\left(H_{ap}\left(e^{j\omega}\right)\right)$

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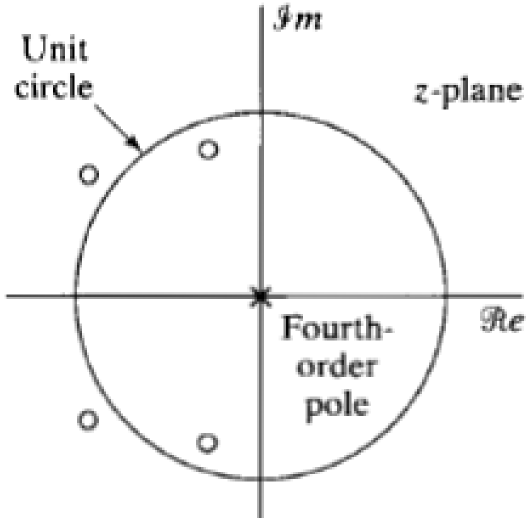
Minimum Phase Systems

Frequency Response Compensation

Example 5.15

$$H_d(z) = (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1})(1 - 1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1})$$

- Since the system contains only zeros, so the system is
  - FIR
  - Stable
  - Causal
- Since two of the zeros are outside the unit circle, the system is
  - Non-minimum phase

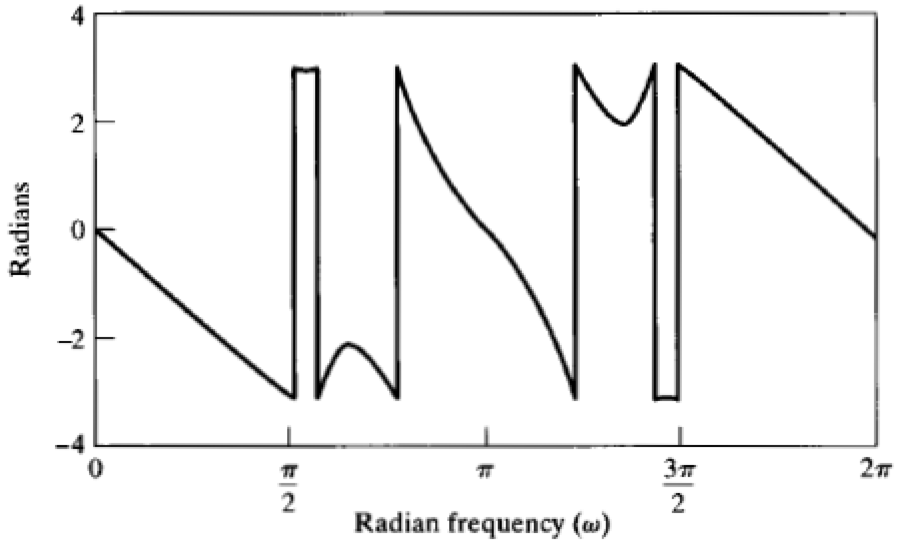


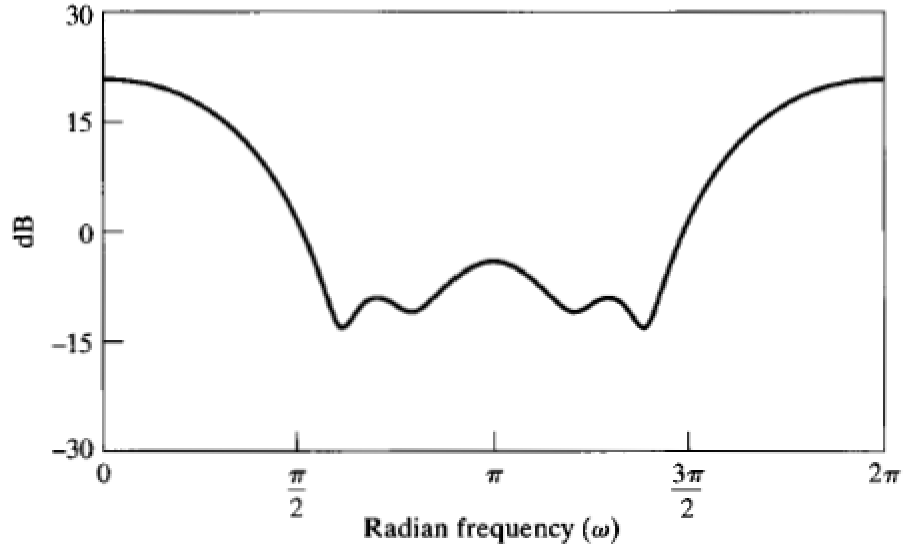
Minimum Phase Systems

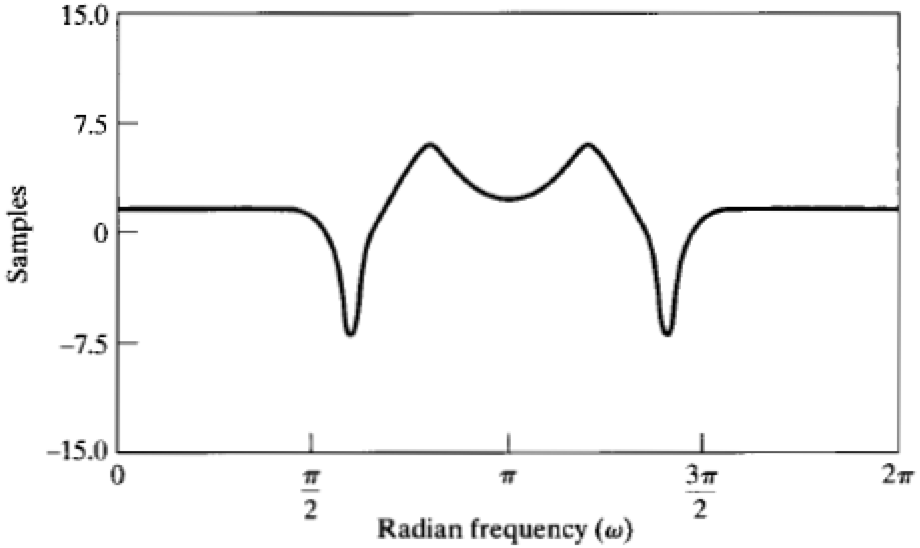
Frequency Response Compensation

Example 5.15

$\boldsymbol{H_d(e^{j\omega})}$







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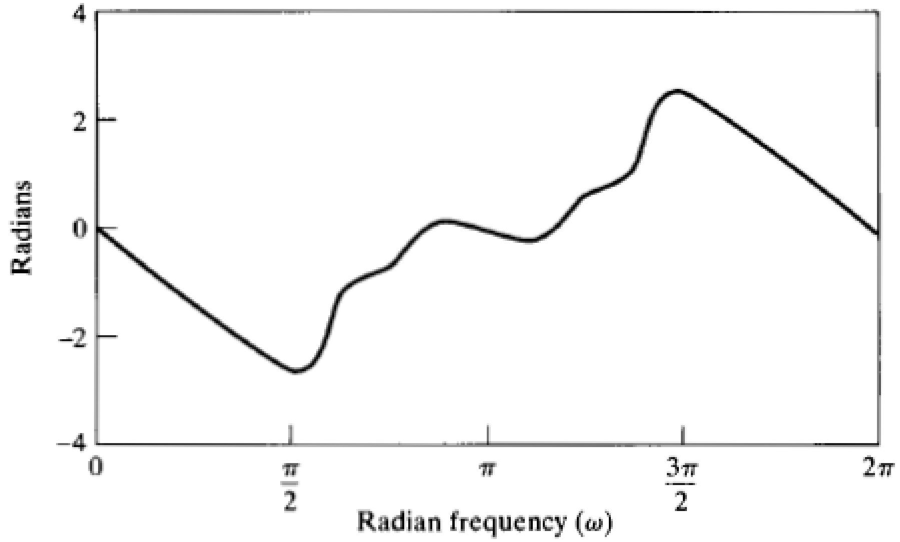
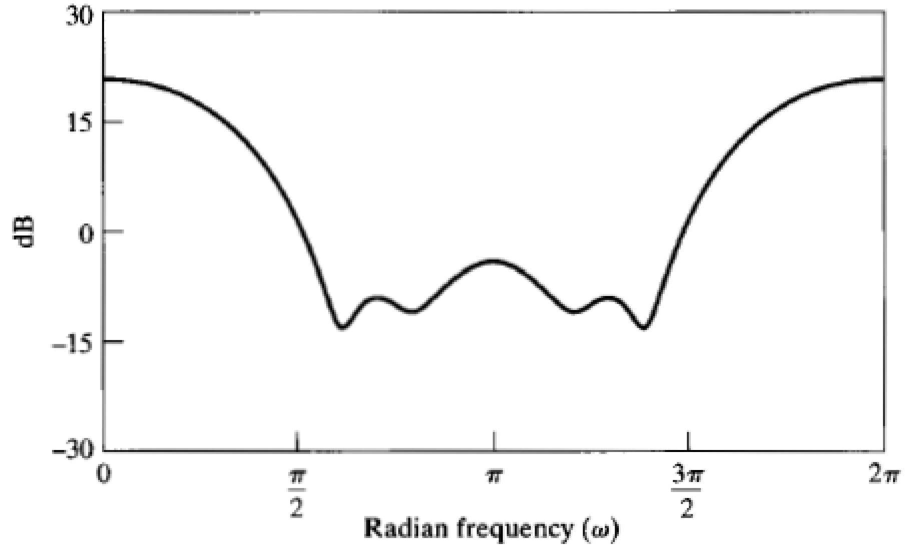
Minimum Phase Systems

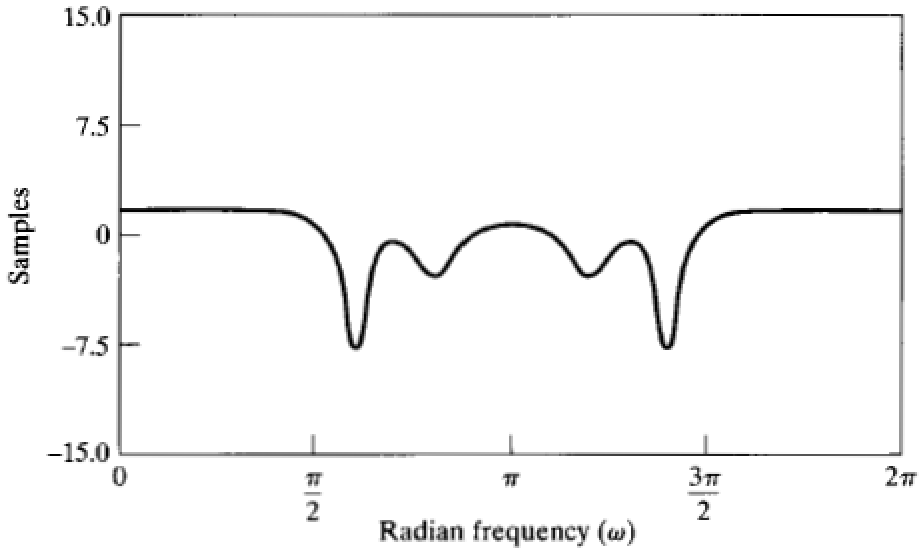
Frequency Response Compensation

Example 5.15

$H_{min}(e^{j\omega})$







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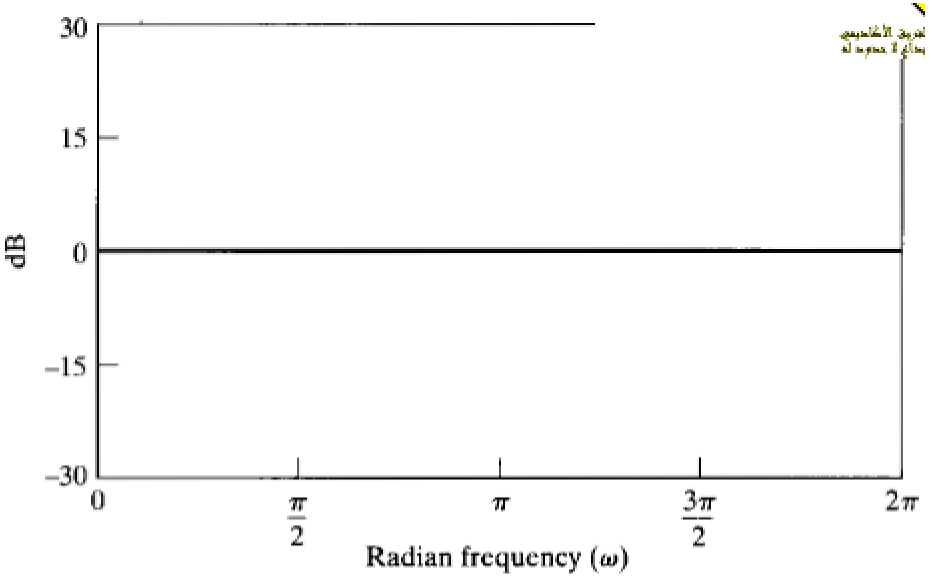
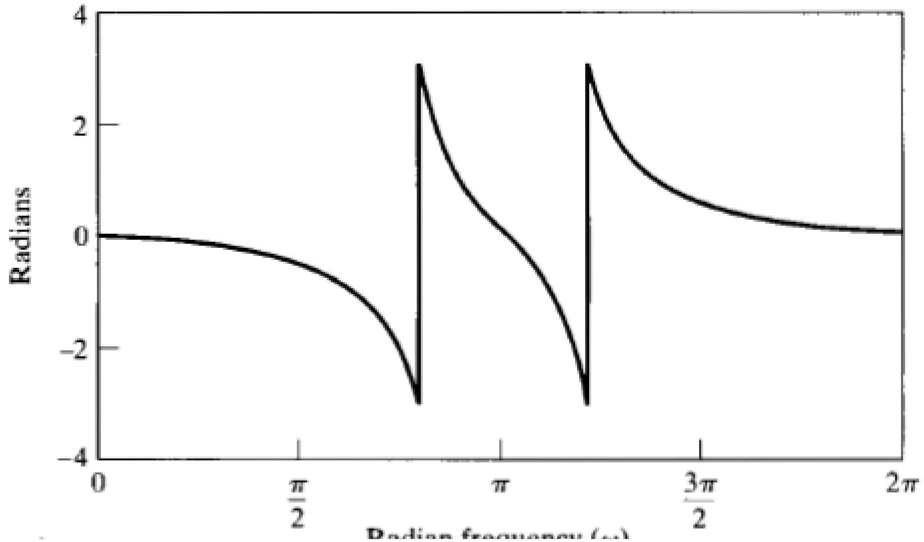
Minimum Phase Systems

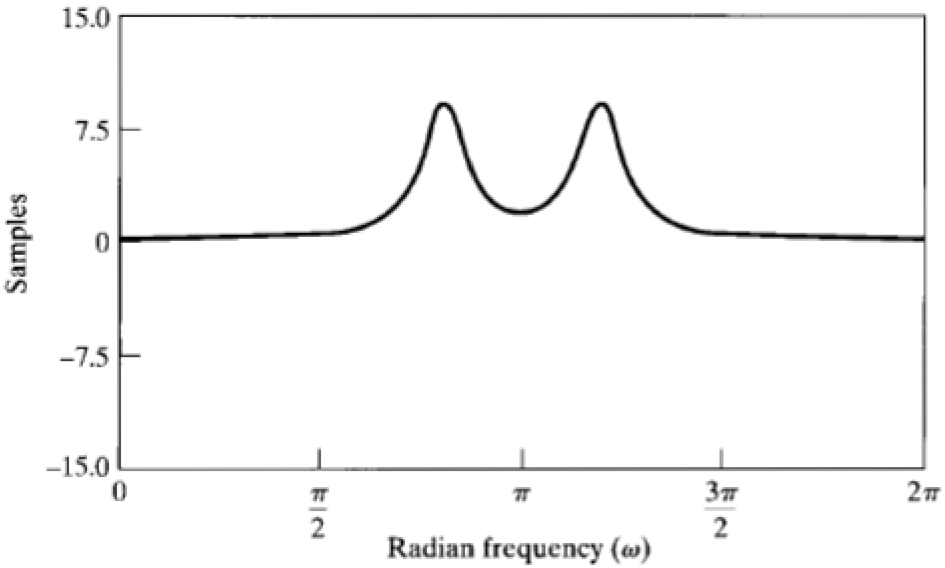
Frequency Response Compensation

Example 5.15

$H_{ap}(e^{j\omega})$







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Minimum Phase Systems

Frequency Response Compensation

Example 5.15

- The inverse system for $H_d(z)$ would have poles at $z = 1.25e^{\pm j0.8\pi}$ and at $z = 0.9e^{\pm j0.6\pi}$ and thus the causal inverse system would be unstable.
- On the other hand, the minimum phase inverse would be the reciprocal of $H_{min}(z)$ and if this inverse were used in the cascade system of Fig. 5.25, the overall effective system function would be $H_{ap}(z)$.

Minimum Phase Systems

Properties of MPS

- **Minimum Phase Lag**

Since

$$H(z) = H_{min}(z)H_{ap}(z)$$

- So the continuous phase (CP) of any non-minimum phase system can be expressed as

$$\arg[H(e^{j\omega})] = \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

- Therefore, the CP that would correspond to the PV of $H(e^{j\omega})$ is the sum of the CP associated with the MPS and the CP of the APS associated with the PV.

Minimum Phase Systems

Properties of MPS

- **Minimum Phase Lag**
- As the CP of an APS is negative for $0 \leq \omega \leq \pi$,
- So, any non-minimum phase system will have a more negative phase compared to the minimum phase system.
- The negative of the phase is called the phase lag function.
- Hence, minimum phase systems have minimum phase lag and hence are called minimum phase lag systems or in short minimum phase systems.

Minimum Phase Systems

<u>Properties of MPS</u>

- **<u>Minimum Group Delay</u>**

$$\mathrm{g}rd\left[H\left(e^{j\omega}\right)\right] = \mathrm{g}rd\left[H_{min}\left(e^{j\omega}\right)\right] + \mathrm{g}rd\left[H_{ap}\left(e^{j\omega}\right)\right]$$

- The GD of an MPS is always less than the GD of a non-MPS. Because
  - APS has a positive GD (for all values of ω)
- Thus if we consider all the systems that have a given MR, the one that has all its poles and zeros inside the unit circle has the minimum GD.
- Hence, MPS can also be called MGDS.

Minimum Phase Systems

Properties of MPS

- **Minimum Energy Delay**

Four possible systems corresponding to the same MR.

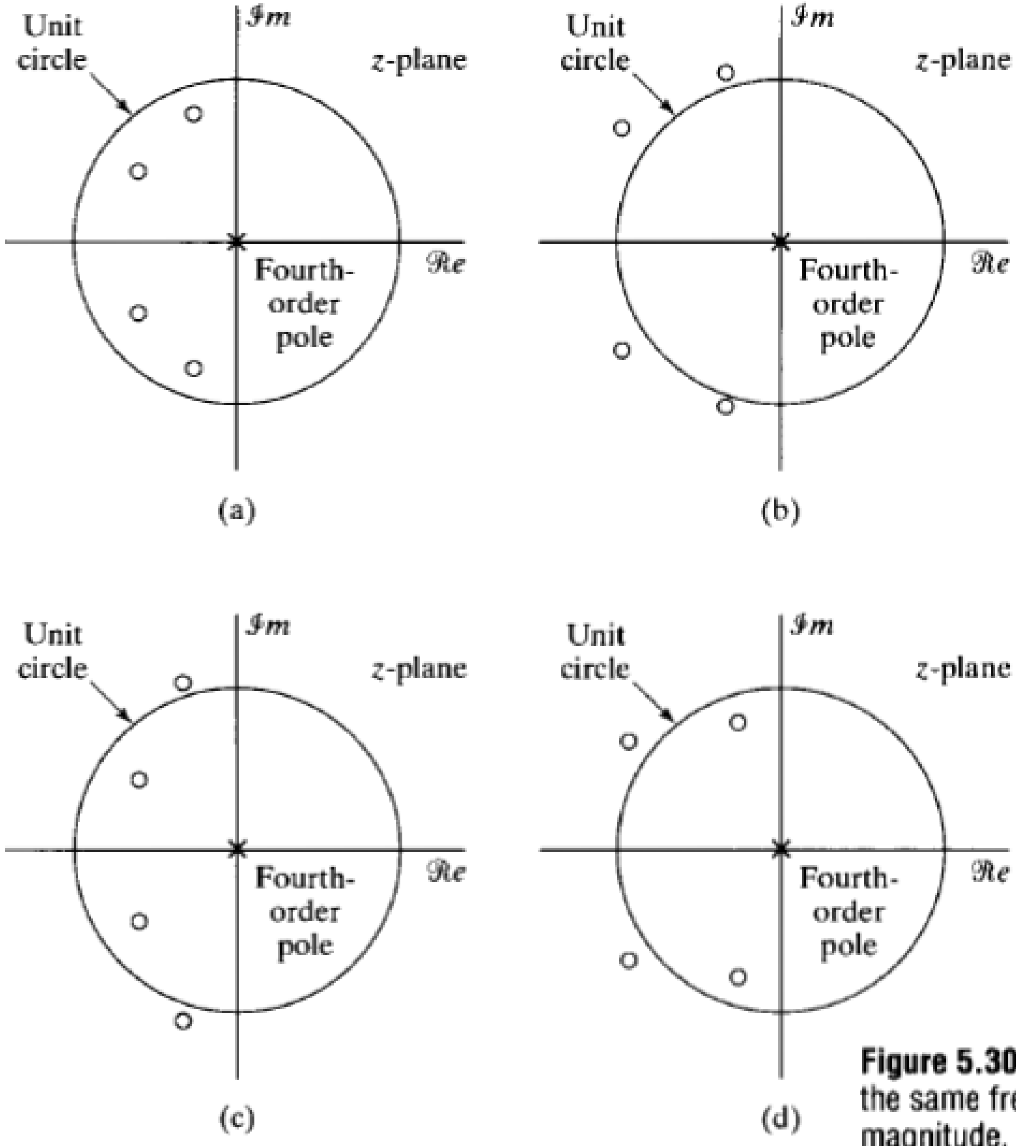

Figure 5.30 Four systems, all having the same frequency-response magnitude. Zeros are at all combinations of $0.9e^{\pm j0.6\pi}$ and $0.8e^{\pm j0.8\pi}$ and their reciprocals.

Minimum Phase Systems

Properties of MPS

- **Minimum Energy Delay**

The associated IRs.

- The IR of the MPS appears to have larger values at the LHS compared to all other systems.
- Hence, MPS concentrate energy in the early part.

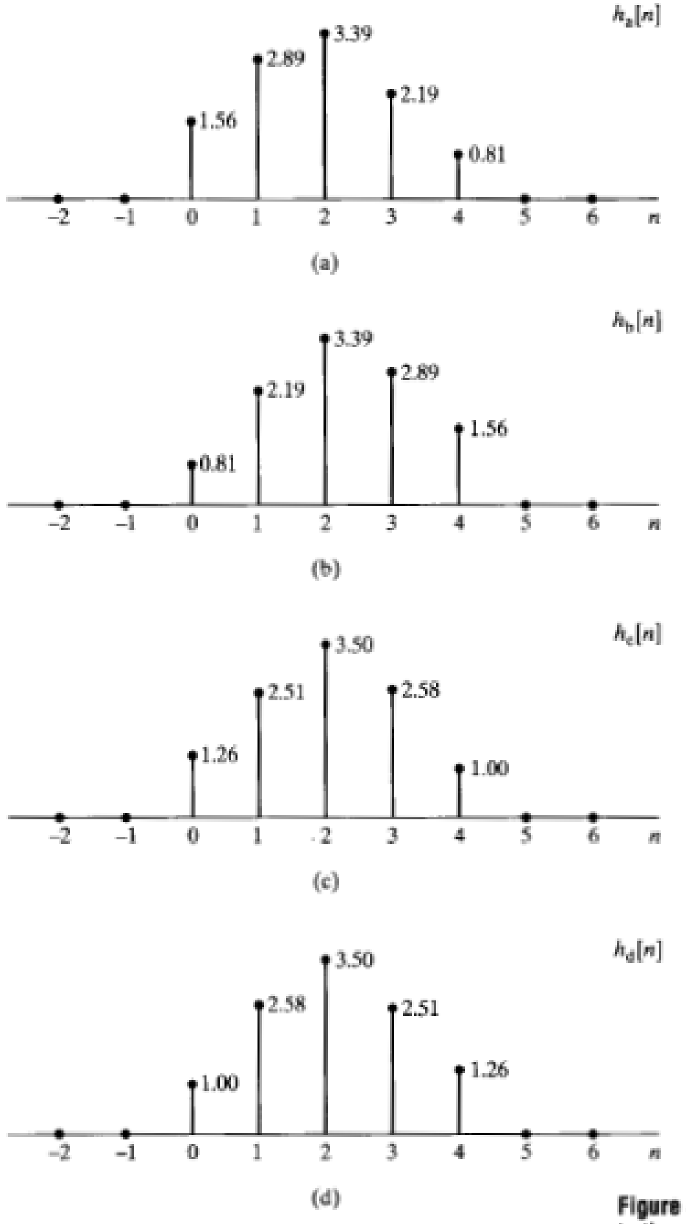

Figure 5.31 Sequences corresponding to the pole–zero plots of Figure 5.30.

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Minimum Phase Systems

<u>Properties of MPS</u>

- **<u>Minimum Energy Delay</u>**
- Of all the systems that have the same MR, the energy of the MPS is delayed the least.
- Hence, MPS are also called MEDS or simply MDS.
  - Minimum energy delay occurs for systems with all its zeros inside the unit circle.
  - Maximum energy delay occurs for systems with all its zeros outside the unit circle.
    - They are also called maximum phase systems.

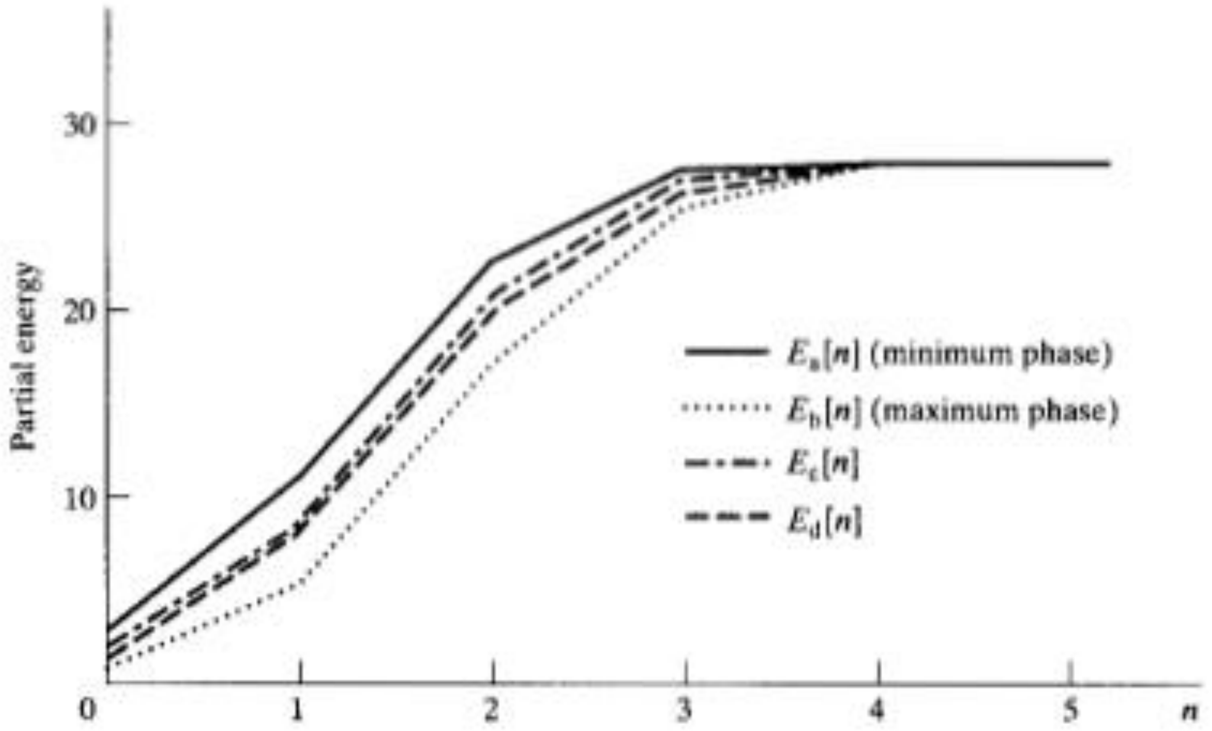

Figure 5.32 Partial energies for the four sequences of Figure 5.31. (Note that $E_a[n]$ is for the minimum-phase sequence $h_a[n]$ and $E_b[n]$ is for the maximum-phase sequence $h_b[n]$.)

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