EEE 324 Digital Signal Processing

Lecture 19

Linear Systems with Generalized Linear Phase

Dr. Shadan Khattak

Department of Electrical Engineering

COMSATS Institute of Information Technology - Abbottabad

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- Systems with Linear Phase
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Linear Systems with Generalized Linear Phase

- For frequency selective filters, it is desirable to have
  - Constant MR at the band of interest.
  - Zero phase at the band of interest.
- For causal systems, zero phase is not achievable.
  - Hence, some phase distortion must be allowed.
  - The effect of linear phase with integer slope is a simple time shift.
  - A non-linear phase can have a major effect on the shape of a signal even when the MR is constant.
  - Hence, it is particularly desirable to design systems to have exactly or approximately linear phase.
  - A filter needs linear phase to in order to properly shift every cosine by the same amount.
  - This constraint is stricter than what we actually need for most filters.
  - In this lecture, we consider formalization and generalization of the notions of linear phase and ideal time delay by considering the class of systems that have constant group delay.

Linear Systems with Generalized Linear Phase

<u>Systems with Linear Phase</u>

- Consider an LTI system whose FR over one period is

$$H_{id}(e^{j\omega}) = e^{-j\omega\alpha}, \qquad |\omega| < \pi \qquad (1)$$

Where α is a real number (not necessarily integer)

- Such a system is an ideal delay system with a delay of α.
- This type of system has:
  - Constant MR ($\left|H_{id}(e^{j\omega})\right| = 1$)
  - Linear PR ($angle\left(H_{id}(e^{j\omega})\right) = -\omega\alpha$)
  - Constant GD ($grd\left[H_{id}(e^{j\omega})\right] = \alpha$)

Linear Systems with Generalized Linear Phase

Systems with Linear Phase

The inverse FT of $H_{id}\left(e^{j\omega}\right)$ is the impulse response

$$h_{id}[n] = \frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)}, \qquad -\infty < n < \infty$$

The O/P of this system for an I/P $x[n]$ is

$$y[n] = x[n] * \frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)} = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin\pi(n-k-\alpha)}{\pi(n-k-\alpha)}$$

If $\alpha = n_d$ is an integer, then

$$h_{id}[n] = \delta[n-n_d] \text{ and}$$
$$y[n] = x[n] * \delta[n-n_d] = x[n-n_d]$$

Linear Systems with Generalized Linear Phase

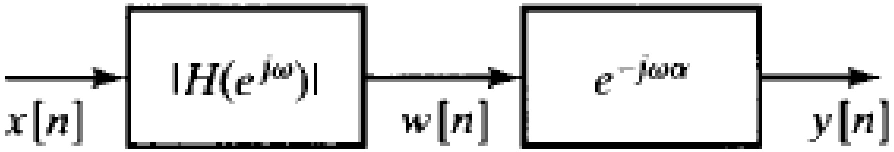
Systems with Linear Phase

- If the GD α is positive,
  - The time shift is a time delay.
- If the GD α is negative,
  - The time shift is a time advance.

Linear Systems with Generalized Linear Phase

Systems with Linear Phase

Representation of an LP LTI system as a cascade of a magnitude filter and time shift.



$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha} \qquad (2)$$

- The signal $x[n]$ is filtered by the zero phase FR $|H(e^{j\omega})|$ and the filtered output is then time-shifted by the amount α.

Linear Systems with Generalized Linear Phase

Systems with Linear Phase

Representation of an LP LTI system as a cascade of a magnitude filter and time shift.

If $H(e^{j\omega})$ is a linear phase ideal low pass filter,

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

The corresponding impulse response is

$$h_{lp}[n] = \frac{sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

Linear Systems with Generalized Linear Phase

Systems with Linear Phase

Example 5.16 (Ideal Low Pass with Linear Phase)

$\omega_c = 0.4\pi$

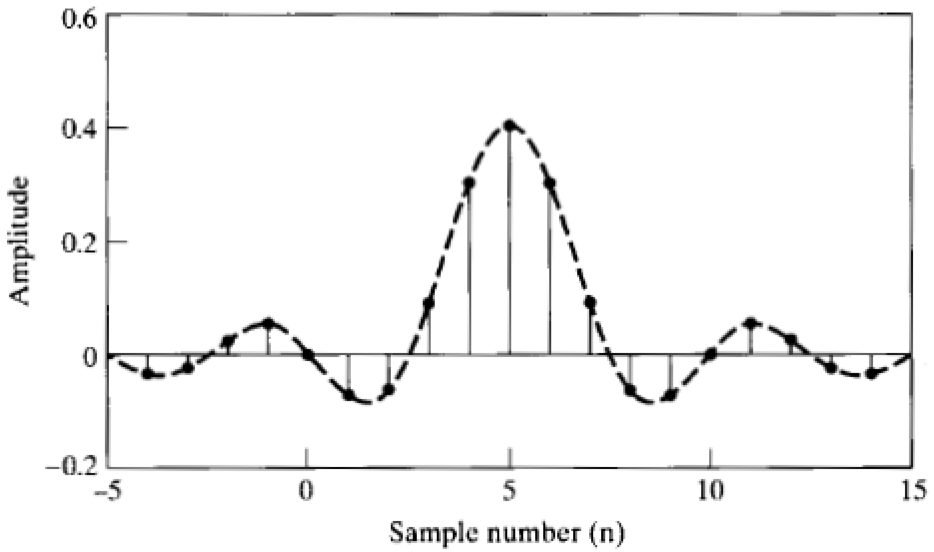
$\alpha = n_d = 5$

When α is an integer, the IR is symmetric about $n = n_d$

$$h_{lp}[2n_d - n] = \frac{sin\omega_c(2n_d - n - n_d)}{\pi(2n_d - n - n_d)}$$

$$= \frac{sin\omega_c(n_d - n)}{\pi(n_d - n)}$$

$$= h_{lp}[n]$$



Dr. Shadan Khattak
Department of Electrical Engineering
CIIT - Abbottabad

Linear Systems with Generalized Linear Phase

Systems with Linear Phase

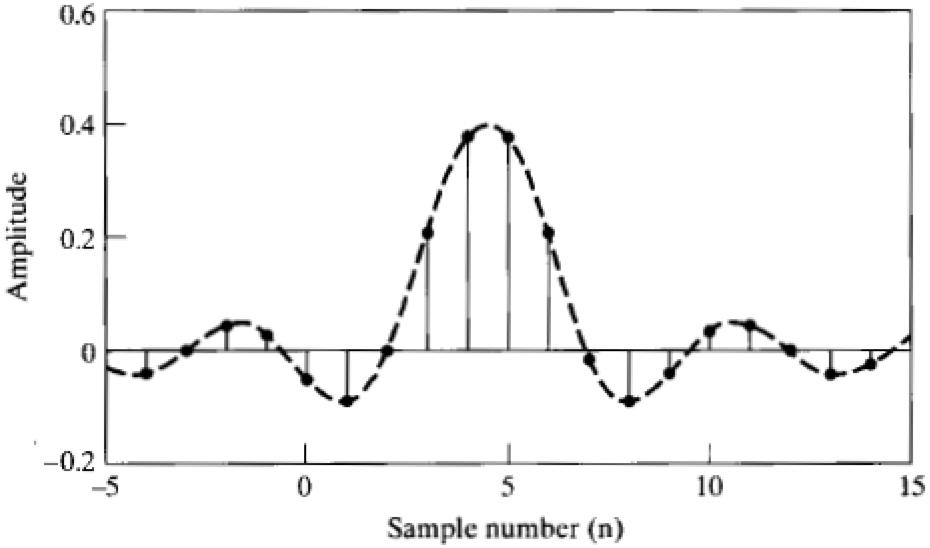
Example 5.16 (Ideal Low Pass with Linear Phase)

$\omega_c = 0.4\pi$

$\alpha = n_d = 4.5$

Symmetry around $\alpha = n_d = 4.5$ which is a non-integer.

If 2α is an integer, then $h_{lp}[2\alpha - n] = h_{lp}[n]$



Linear Systems with Generalized Linear Phase

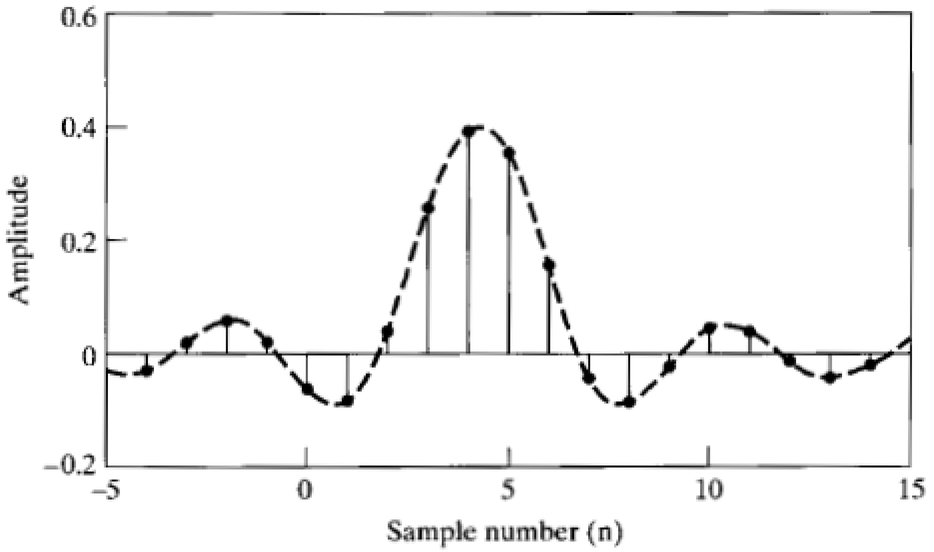
Systems with Linear Phase

Example 5.16 (Ideal Low Pass with Linear Phase)

$\omega_c = 0.4\pi$

$\alpha = n_d = 4.3$

No symmetry at all



Linear Systems with Generalized Linear Phase

Systems with Linear Phase

In general, an LPS has FR

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha} \qquad (2)$$

- If 2α is an integer (i.e., if α is an integer or an integer plus one-half), the corresponding IR has even symmetry about α i.e.,

$$h[2\alpha - n] = h[n]$$

- If 2α is not an integer, then the IR will not have even symmetry, but it will still have linear phase and constant GD.

Linear Systems with Generalized Linear Phase

Generalized Linear Phase

- We don't always need linear filters.
- In the earlier discussion on LPS, we discussed a class of systems whose FR is of the form of Eq. (2) i.e.,
  - A real valued non-negative function of ω multiplied by a linear phase term $e^{-j\omega\alpha}$.
- It is possible to generalize the definition of linear phase.

Linear Systems with Generalized Linear Phase

Generalized Linear Phase

- A system is called a generalized linear phase system if its FR can be expressed in the form

$$H\left(e^{j\omega}\right) = A(e^{j\omega})e^{-j\alpha\omega+j\beta} \qquad \textbf{(3)}$$

where α and β are constants and $A(e^{j\omega})$ is a real (possibly bipolar) function of ω.

- For such systems, the phase consists of constant terms added to the linear function $-\omega\alpha$ i.e., $-\omega\alpha + \beta$ is the equation of a straight line.
- If we ignore the discontinuities that result from the addition of constant phase over all or part of the band $|\omega| < \pi$,
  - Then the system can be characterized by constant GD.

Linear Systems with Generalized Linear Phase

Generalized Linear Phase

- Hence, the class of systems such that

$$\tau(\omega) = grd\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\} = \alpha$$

Have linear phase of the more general form

$$\arg\left[H\left(e^{j\omega}\right)\right] = \beta - \omega\alpha$$

Where β and α are both real constants.

Linear Systems with Generalized Linear Phase

Generalized Linear Phase

- Earlier, we noted that the IR of LP systems may have symmetry about α if 2α is an integer.
- For systems with GLP,

$$H(e^{j\omega}) = A(e^{j\omega})e^{j(\beta-\alpha\omega)}$$

Necessary (but not sufficient) condition for the system to have constant delay:

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha)+\beta] = 0 \; for\ all\ \omega \qquad (4)$$

Linear Systems with Generalized Linear Phase

Generalized Linear Phase

E.g.,

One set of conditions that satisfies Eq. (4) is:

- $\beta = 0$ or π
- $2\alpha = M = an\ integer$
- $h[2\alpha - n] = h[n]$ (5)

$\Rightarrow A(e^{j\omega})$ is an even function of ω

Linear Systems with Generalized Linear Phase

<u>Generalized Linear Phase</u>

<u>Another set of conditions that satisfies Eq. (4) is:</u>

- $\beta = 3\pi/2$ or $\pi/2$
- $2\alpha = M = an\ integer$
- $h[2\alpha - n] = -h[n]$ (6)

$\Rightarrow A(e^{j\omega})$ is an odd function of ω

Linear Systems with Generalized Linear Phase

Generalized Linear Phase

- These two set of conditions, guarantee GLP or CGD, but there are other systems that also satisfy Eq. (3) without these symmetry conditions.