EEE 324 Digital Signal Processing

# Lecture 1

***Periodic Sampling,***
***Frequency Domain Representation of Sampling***

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# 1. Periodic Sampling

- A DT sequence $x[n]$ is obtained from a CT signal $x_c(t)$ according to the relation:

$$\boxed{x[n] = x_c(nT) \qquad\qquad -\infty < n < \infty} \qquad\qquad (1)$$

$$T: sampling\ period$$

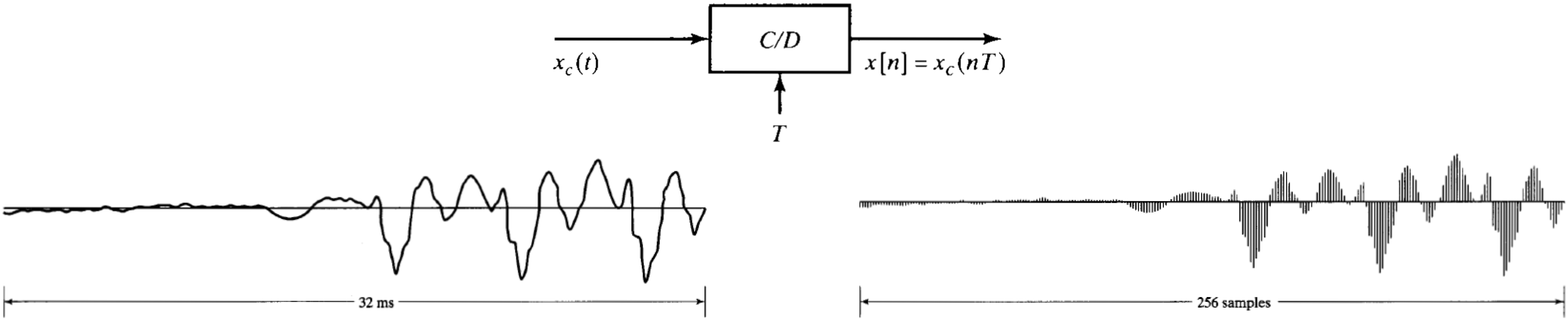
$$\frac{1}{T} = f_s: sampling\ frequency \text{ (samples/sec.)}$$

$$\Omega_s = \frac{2\pi}{T}: sampling\ frequency \text{ (radians/sec.)}$$

# Periodic Sampling

**<u>An ideal Continuous-to-Discrete (C/D) converter</u>**

- A system which implements Eq. (1).





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# Periodic Sampling

- Practically, sampling is achieved using Analog-to-Digital (A/D) converter.
- Similar to C/D converter but also includes quantization.

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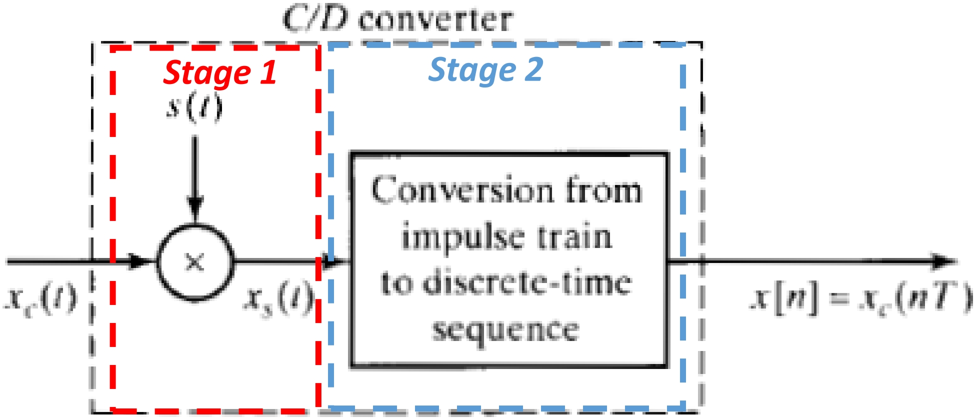
# Periodic Sampling

- Sampling is, generally, not invertible.
- Mathematically, sampling can be understood as a two-stage process:
- ***Stage 1:*** Impulse train modulation
- ***Stage 2:*** Conversion of the impulse train to a sequence



# Periodic Sampling

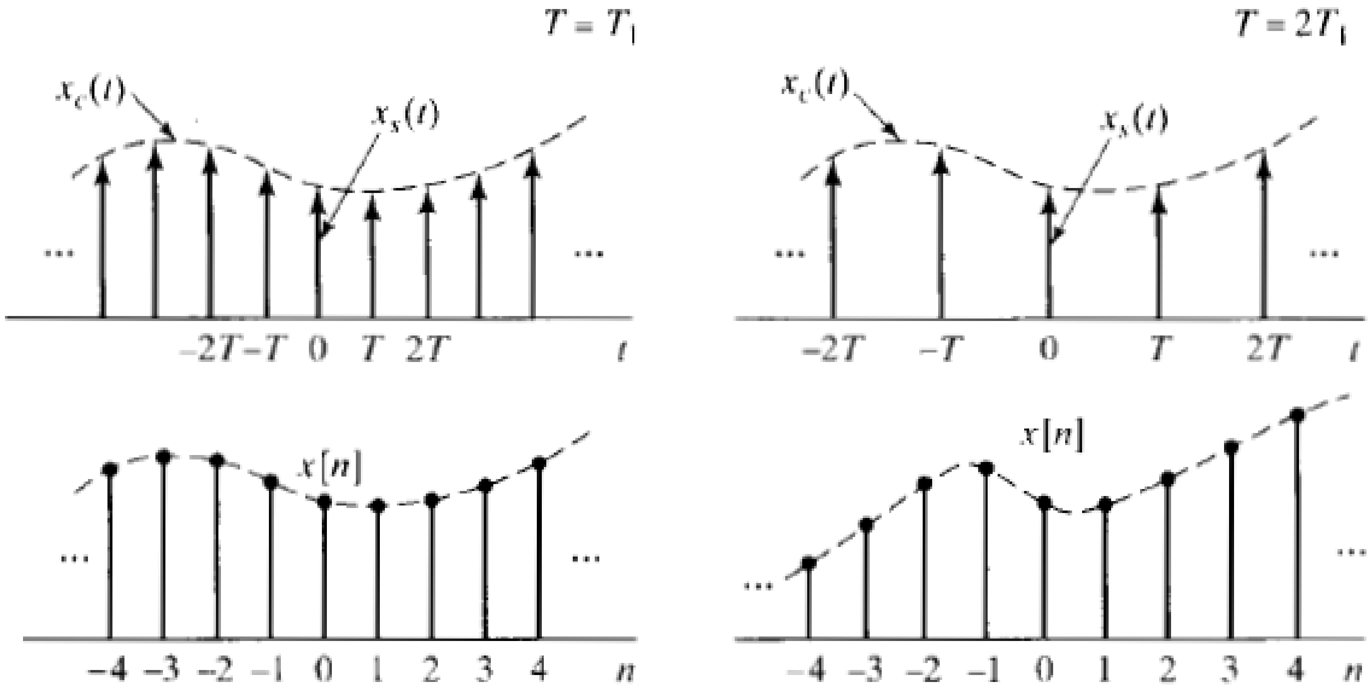
## Sampling using two different sampling rates

Impulse Train Sampling

$(x_c(t) \rightarrow x_s(t))$

Impulse Train to DT Seq.

$(x_s(t) \rightarrow x[n])$



# Periodic Sampling

**<u>Difference between $\boldsymbol{x_s(t)}$ and $\boldsymbol{x[n]}$</u>**

- $x_s(t)$ is CT while $x[n]$ is DT.
- $x_s(t)$ includes time information while $x[n]$ does not.
- The samples of $x_s(t)$ are represented as the areas of impulses while those of $x[n]$ are finite numbers.

# Frequency Domain Representation of Sampling

- In the next few slides, we will see how we can obtain $x_s(t)$ from $x_c(t)$.

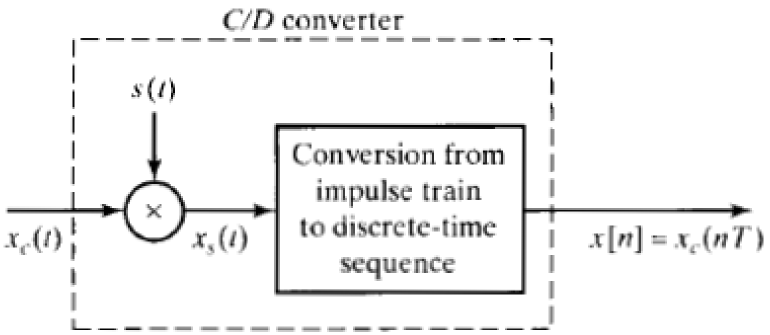
**Conversion from $x_c(t)$ to $x_s(t)$**

The original CT signal: $x_c(t)$

The CT impulse train: $s(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$

Result of modulation: $\boxed{x_s(t) = x_c(t)s(t)}$ (2)

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$




Using the sifting property of the impulse

i.e., $\emptyset(t)\,\delta(t - t_0) = \emptyset(t_0)\,\delta(t - t_0)$

$\boxed{x_s(t) = \sum_{-\infty}^{\infty} x_c(nT)\delta(t - nT)}$ (3)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# Frequency Domain Representation of Sampling

**Conversion from $x_c(t)$ to $x_s(t)$**

CTFT of $x_s(t) = X_s(j\Omega)$

CTFT of $x_c(t) = X_c(j\Omega)$

CTFT of $s(t) = S(j\Omega)$

*The CTFT of a periodic impulse train is a periodic impulse train i.e.,*

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{CTFT}{\longleftrightarrow} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

*So,*

$$\boxed{S(j\Omega) = \frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta(\Omega - k\Omega_s)} \quad (4)$$

# Frequency Domain Representation of Sampling

**Conversion from $x_c(t)$ to $x_s(t)$**

Since

$$x_s(t) = x_c(t)s(t),$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$\boxed{X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))} \quad (5)$$

Eq. (5) shows the relationship between the FT of the input and output of Stage 1.

- FT of the output consists of periodically repeated copies of the FT of the input.
- Each copy is shifted by integer multiples of the sampling frequency and then superimposed on each other.

# Frequency Domain Representation of Sampling

## Conversion from $x_c(t)$ to $x_s(t)$

Example:

Case 1: $\Omega_s > 2\Omega_N$

⇒ The replicas of $X_c(j\Omega)$ do not overlap

⇒ When the replicas of $X_c(j\Omega)$ are added together, there remains (within a scale factor of 1/T) a replica of $X_c(j\Omega)$ at each integer multiple of $\Omega_s$.

⇒ $x_c(t)$ can be recovered from $x_s(t)$ with an ideal low pass filter.

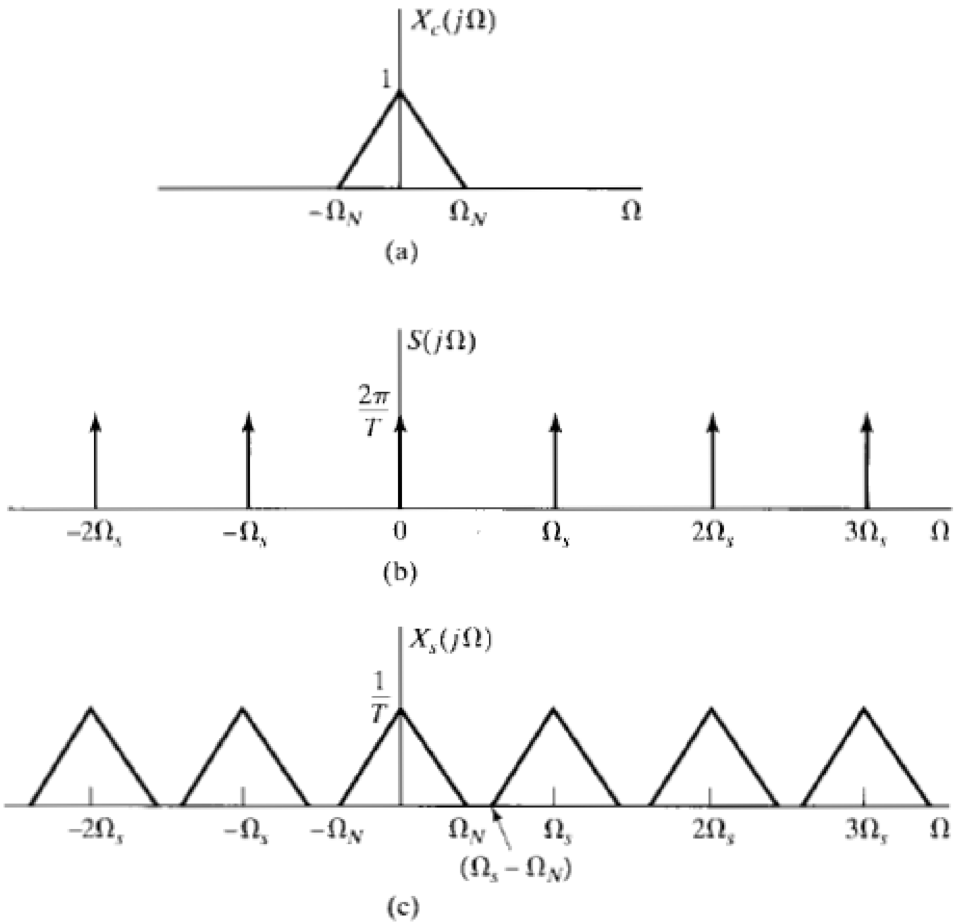



**Figure 4.3** Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

## Frequency Domain Representation of Sampling

**Conversion from $x_c(t)$ to $x_s(t)$**

Example:

Case 2: $\Omega_s < 2\Omega_N$

$\Rightarrow$ The replicas of $X_c(j\Omega)$ overlap

$\Rightarrow$ When the replicas of $X_c(j\Omega)$ are added together, $X_c(j\Omega)$ is no longer recoverable by an ideal low pass filter.

$\Rightarrow$ The reconstructed signal $x_r(t)$ is an alias of the original signal $x_c(t)$.

$\Rightarrow$ There exists aliasing distortion between $x_r(t)$ and $x_c(t)$.

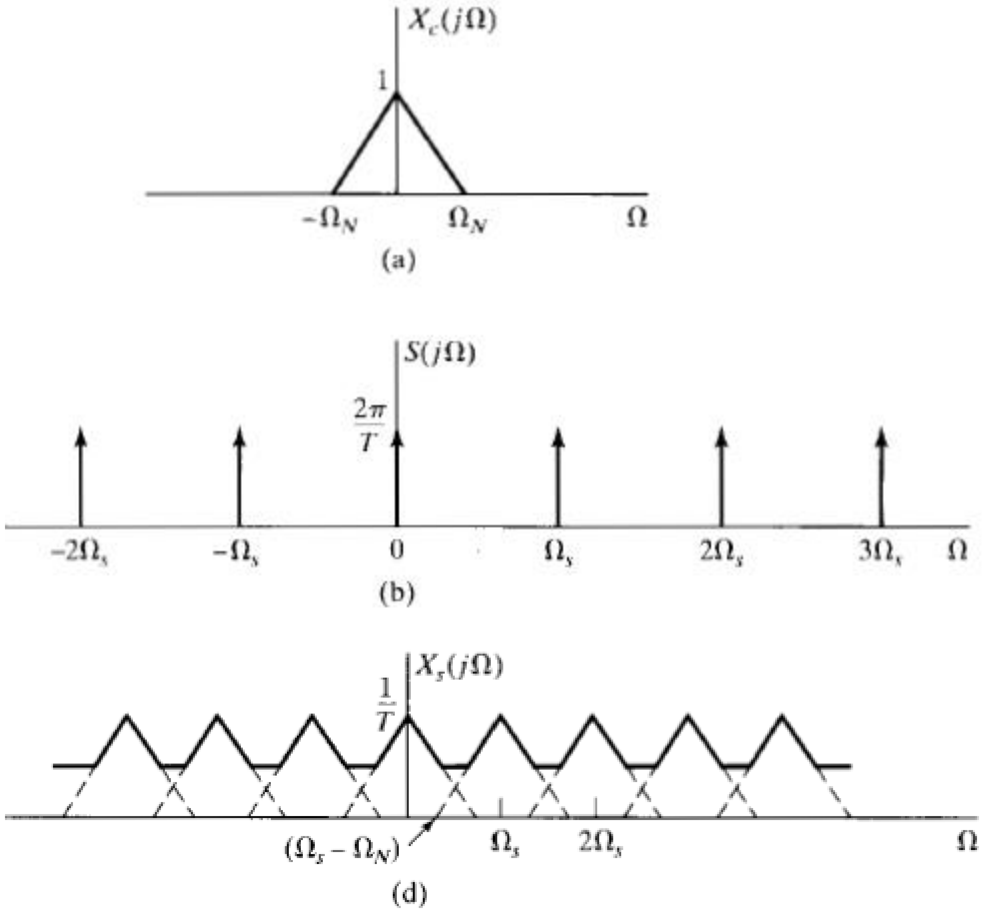



**Figure 4.3** Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

# Frequency Domain Representation of Sampling

**Recovering $x_c(t)$ from $x_s(t)$**

Case 1: $\Omega_s > 2\Omega_N$

Example:

$H_r(j\Omega)$ is a low pass reconstruction filter with gain $T$ and cut-off frequency $\Omega_c$.

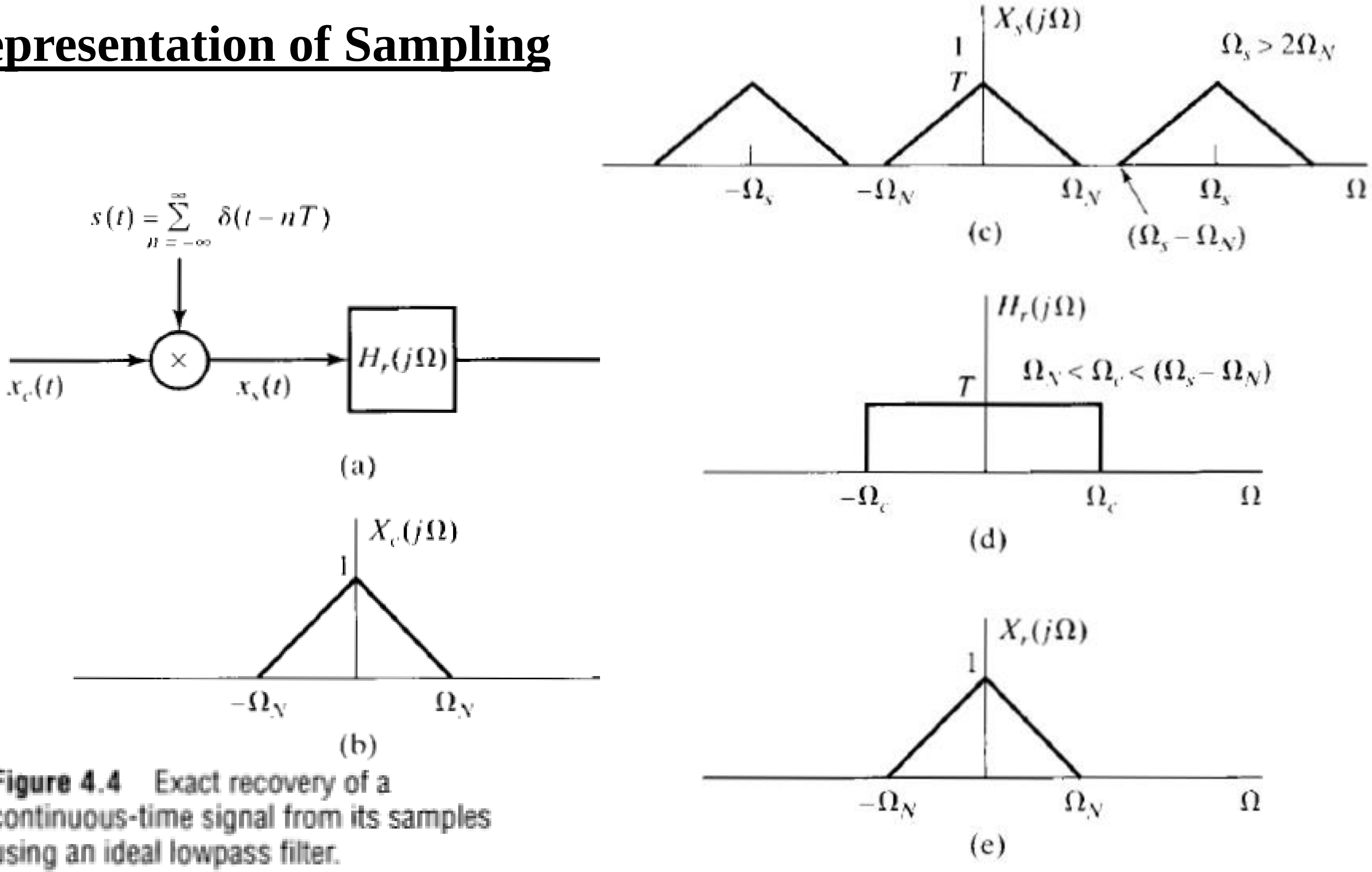



**Figure 4.4** Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

# Frequency Domain Representation of Sampling

## Recovering $x_c(t)$ from $x_s(t)$

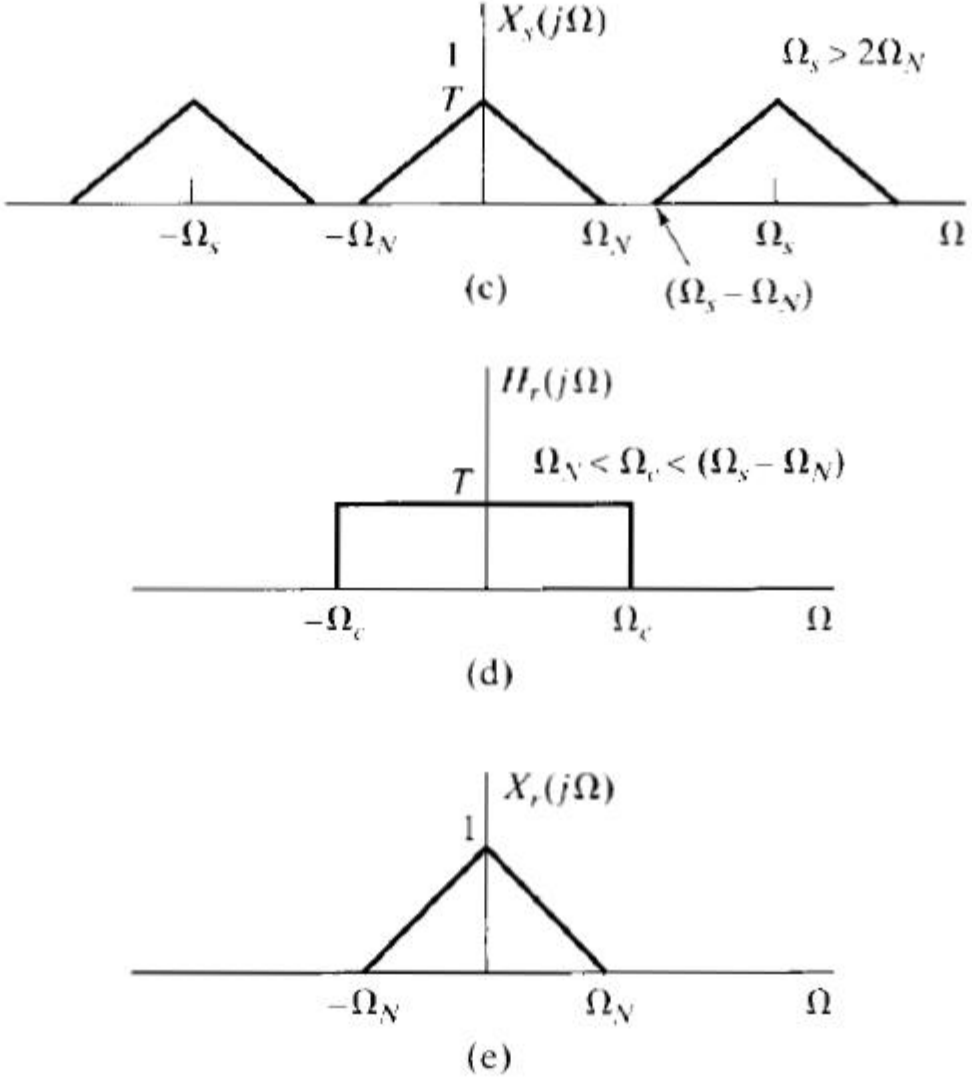
Case 1: $\Omega_s > 2\Omega_N$

- $H_r(j\Omega)$ is a low pass reconstruction filter with gain $T$ and cut-off frequency $\Omega_c$.

$$X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega)$$

Such that $\Omega_N < \Omega_c < (\Omega_s - \Omega_N)$

Then $X_r(j\Omega) = X_c(j\Omega)$



# Frequency Domain Representation of Sampling

**Recovering $x_c(t)$ from $x_s(t)$**

Case 2: $\Omega_s < 2\Omega_N$ (The case of aliasing)

$$X_r(j\Omega) \neq X_c(j\Omega)$$

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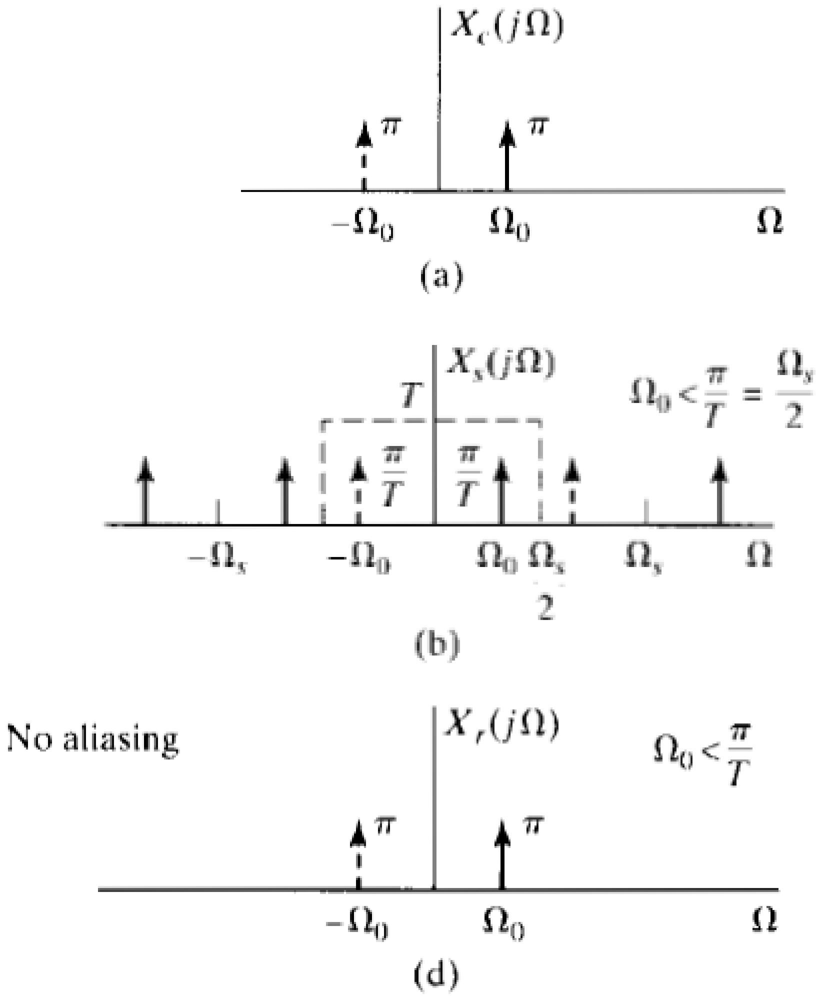
# Frequency Domain Representation of Sampling

**Recovering $x_c(t)$ from $x_s(t)$**

Example: $x_c(t) = cos\Omega_0 t$

$$\Omega_0 < \frac{\Omega_s}{2}$$

The reconstructed output $x_r(t) = cos\Omega_0 t$



# Frequency Domain Representation of Sampling

## Recovering $x_c(t)$ from $x_s(t)$

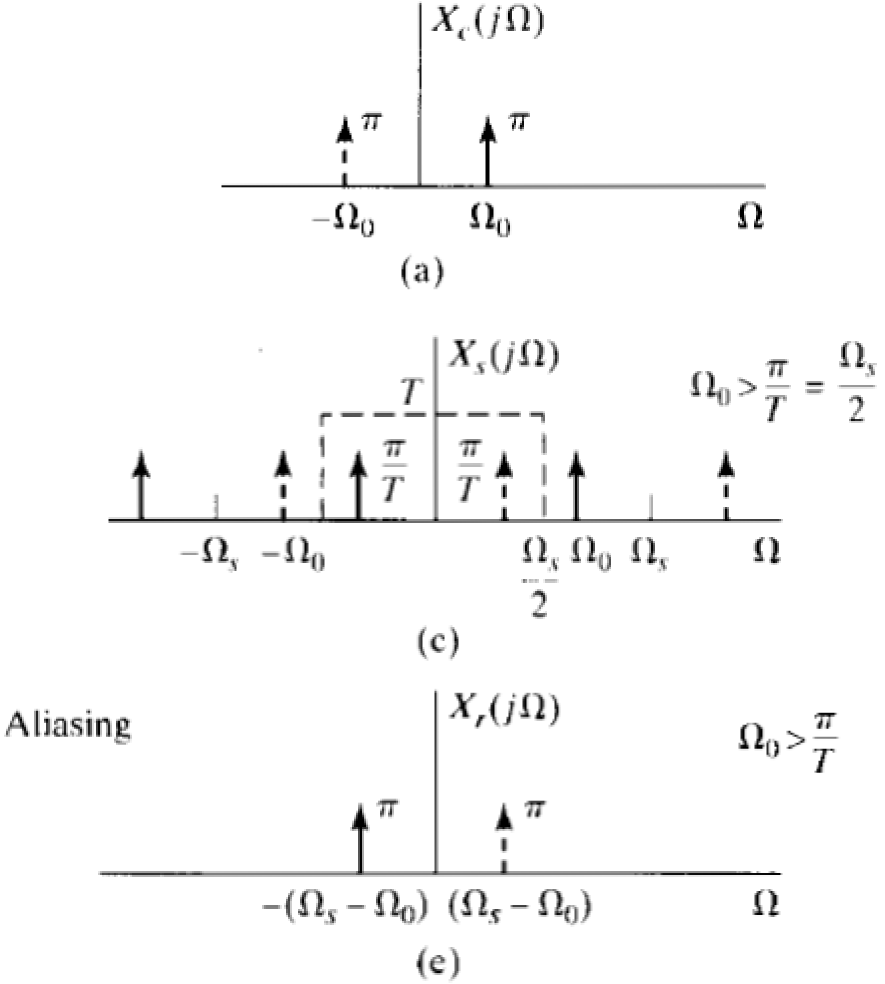
Example: $x_c(t) = cos\Omega_0 t$

$$\Omega_0 < \frac{\Omega_s}{2}$$

The reconstructed output is:

$$x_r(t) = \cos(\Omega_s - \Omega_0)\, t$$

- The higher frequency signal $cos\Omega_0 t$ has taken on the identity (alias) of the lower frequency signal $\cos(\Omega_s - \Omega_0)\, t$





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# Frequency Domain Representation of Sampling

**Nyquist Sampling Theorem**

Let $x_c(t)$ be a band-limited signal with

$$X_c(j\Omega) = 0 \; for \; |\Omega| \geq \Omega_N$$

Then

$x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), \qquad n = 0, \pm 1, \pm 2, \ldots \quad if$$

$$\boxed{\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N} \qquad (7)$$

$\Omega_N$ is referred to as the ***Nyquist frequency.***

$2\Omega_N$ is referred to as the ***Nyquist rate.***

***Physical meaning of Nyquist theorem is that to recover $x_c(t)$ from its samples $x[n] = x_c(nT)$, there must be at least 2 samples present within each cycle.***

# Frequency Domain Representation of Sampling

**Expressing $X(e^{j\omega})$ in terms of $X_s(j\Omega)$ and $X_c(j\Omega)$**

Applying CTFT to Eq. (3).

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}$$

Since

$$x[n] = x_c(nT)$$

And

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

So,

$$\boxed{X_s(j\Omega) = X(e^{j\omega})\big|_{\omega=\Omega T} = X\left(e^{j\Omega T}\right)} \qquad (8)$$

Equivalently,

$$\boxed{X\left(e^{j\Omega T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))} \qquad (9)$$

And

$$\boxed{X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)} \qquad (10)$$

# Frequency Domain Representation of Sampling

**Expressing $X(e^{j\omega})$ in terms of $X_s(j\Omega)$ and $X_c(j\Omega)$**

From Eqs. (8) - (10),

- $X(e^{j\omega})$ is a frequency scaled version of $X_s(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$.

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# Take Home!

- $x_c(t)$ can be uniquely determined by its samples $x[n]\ \ if\ \Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$
- The FT of the output of the sampler is consists of the periodically repeated and amplitude scaled replicas of the FT of the input.

# Reading

- Section 4.0 – 4.2 (Oppenheim)

# Practice Problems

- Problems 4.1 – 4.4, 4.8 – 4.11 (Oppenheim)