Carnegie Mellon DSP-I Electrical & Computer ENGINEERING

Digital Signal Processing I (18-791)
Fall Semester, 2005

IIR FILTER DESIGN EXAMPLE

These notes summarize the design procedure for IIR filters as discussed in class on November 2.

Introduction:

We are discussing how to design a discrete-time lowpass filter using both the impulse invariance method and the bilinear transform method. The specs for the filter to be designed are

- Passband cutoff frequency: $\omega_p = 0.15\pi$
- Stopband cutoff frequency: $\omega_s = 0.35\pi$
- Passband ripple[1]: $-3 \text{ dB} \le \left|H(e^{j\omega})\right| \le 0 \text{ dB}, \ |\omega| \le \omega_p$
- Stopband attenuation: $\left|H(e^{j\omega})\right| \le -20 \text{ dB}, \ \omega_s \le |\omega| \le \pi$

In terms of the parameters δ_1 and δ_2, this means that

$$1 - \delta_1 = 10^{-3/20} = 0.7079$$

$$\delta_2 = 10^{-20/20} = 0.1$$

For convenience we define two new parameters k_1 and k_2 that are closely related to δ_1 and δ_2 as follows:

$$k_1 \equiv \frac{1}{(1-\delta_1)^2} - 1 = 0.9953$$

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1. Note that the values of passband and stopband attenuation are given in dB. If x has the dimensions of amplitude or magnitude (as opposed to energy or power), the value of x expressed in dB would be $20\log_{10}(x)$

$$k_2 = \frac{1}{\delta_2^2} - 1 = 99$$

Finally, we are designing our filter by transforming the frequency response of a continuous-time Butterworth filter. As you will recall, the Butterworth filter has the transfer function

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

The parameter N specifies the number of poles in the prototype filter, and the parameter Ω_c specifies the critical cutoff frequency. Note that the magnitude response of the Butterworth filter decreases monotonically with increasing N, that the falloff in response as we transit from passband to stopband becomes sharper as N increases, and that $|H(j\Omega)|^2$ equals 1.0 when $\Omega = 0$ and $1/2$ when $\Omega = \Omega_c$ for all values of N.

As you will recall from class, a Butterworth filter with parameters N and Ω_c has N poles that fall in the left half of the s-plane, in complex conjugate locations along a circle of radius Ω_c separated by angles of π/N.

Design procedure using impulse invariance

Determining design parameters

Using the impulse invariance design procedure, we have noted that the relation between frequency in the continuous-time and discrete-time domains is $\omega = \Omega T$, where T is merely a design parameter (and not necessarily the sampling frequency). Leaving T as an arbitrary constant for now, we obtain

$$\Omega_p = \omega_p / T = 0.47124/T$$

$$\Omega_s = \omega_s / T = 1.0996/T$$

At the passband edge frequency,

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega_p}{\Omega^c}\right)^{2N}} = (1-\delta_1)^2$$

With a few algebraic manipulations we obtain

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{(1-\delta_1)^2} - 1 = k_1$$

Similarly, at the stopband edge frequency we have

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \delta_2{}^2 \text{ which produces}$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \frac{1}{\delta_2^2} - 1 = k_2$$

Dividing, we obtain

$$\frac{(\Omega_p/\Omega_c)^{2N}}{(\Omega_s/\Omega_c)^{2N}} = \frac{k_1}{k_2} \text{ or}$$

$$N = \frac{1}{2}\frac{\log(k_2/k_1)}{\log(\Omega_s/\Omega_p)} = 2.7144$$

Note that since $\frac{\Omega_s}{\Omega_p} = \frac{\omega_s/T}{\omega_p/T} = \frac{\omega_s}{\omega_p}$, the design parameter T has no effect at all on the value of N that is needed. Furthermore, as expected, the value of N specified increases as either δ_1 or δ_2 decreases in magnitude (which implies decreasing the ripple in the passband and/or stopband) or as the ratio Ω_s/Ω_p decreases (which implies a decrease in the width of the transition band).

Since the number of poles must be an integer, we round up to $N = 3$.

Matching the frequency response exactly at passband produces

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = k_1 \text{ or } \Omega_c = \frac{\Omega_p}{k_1^{1/2N}} = \frac{0.4716}{T}$$

(This value is slightly different from the one I gave in class.) If we were instead to match the frequency response at stopband, we would obtain

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = k_2 \text{ or } \Omega_c = \frac{\Omega_s}{k_2^{1/2N}} = \frac{0.5112}{T}$$

In principle, any value of the critical frequency that satisfies $\frac{0.4716}{T} \le \Omega_c \le \frac{0.5112}{T}$ would be valid. As discussed in class, we will choose the value that meets the passband spec exactly, $\Omega_c = 0.4716/T$ because this value leaves the greatest margin for error at the stopband edge. Since the impulse invariance procedure always incurs a certain degree of aliasing, it will be expected that the *actual* response of the filter at the stopband edge will be larger than designed for. Hence, having the stopband edge response be "overdesigned" will leave more room for error to allow for the effects of aliasing.

Prototype filter design

As noted above, the poles of a Butterworth filter lie in the left half of the *s*-plane on a circle of radius Ω_c in complex conjugate pairs separated by an angle of π/N radians. Since there are three poles, this produces a filter with the system function

$$H(s) = \frac{\Omega_c^3}{(s - \Omega_c e^{j\pi})(s - \Omega_c e^{j(\pi + \pi/3)})(s - \Omega_c e^{j(\pi - \pi/3)})}$$

Plugging in the value of $\Omega_c = 0.4716/T$, we obtain

$$H(s) = \frac{\Omega_c^3}{(s - \Omega_c e^{j\pi})(s - \Omega_c e^{j(\pi + \pi/3)})(s - \Omega_c e^{j(\pi - \pi/3)})}$$

and (with the help of MATLAB) we obtain numerically

$$H(s) = \frac{(0.4716/T)^3}{(s + 0.471\dot{6}/T)(s - (-0.236 + 0.408j)/T)(s - (-0.236 - 0.408j)/T)}$$

Conversion to discrete-time form

Using partial fractions we can rewrite the system function of the continuous-time prototype filter as

$$H(s) = \sum_{k=1}^{N} \frac{A_k}{(s - s_k)}$$

where the parameters A_k are the continuous-time residues of the poles s_k.

Using the MATLAB routine `residue` we find the residues for the three poles, producing the transfer function

$$H(s) = \frac{0.4716}{s + 0.4716/T} + \frac{-0.236 - 0.136j}{s - (-0.236 + 0.408j)/T} + \frac{-0.236 + 0.136j}{s - (-0.236 - 0.408j)/T}$$

The corresponding discrete-time filter has the transfer function

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{\left(1 - e^{s_k T} z^{-1}\right)}$$

Note that the poles of the continuous-time filter, s_k are all of the form of a (generally complex) constant divided by the parameter *T*. Since s_k is multiplied by *T* wherever it appears in the equation for $H(z)$ above, the specific value chosen for T has no effect at all on the discrete-time filter that results from the design process. Hence we normally let $T = 1$ for simplicity. This produces the transfer function

$$H(z) = \frac{0.4716}{1 - \exp(-0.4716)z^{-1}} + \frac{-0.236 - 0.136j}{1 - \exp(-0.236 + 0.408j)z^{-1}} + \frac{-0.236 + 0.136j}{1 - \exp(-0.236 - 0.408j)z^{-1}}$$

or

$$H(z) = \frac{0.4716}{1 - 0.624z^{-1}} + \frac{-0.236 - 0.136j}{1 - (0.725 + 0.313j)z^{-1}} + \frac{-0.236 + 0.136j}{1 - (0.725 - 0.313j)z^{-1}}$$

Combing the second and third terms we obtain

$$H(z) = \frac{0.4716}{1 - 0.624z^{-1}} + \frac{-0.472 + 0.341z^{-1}}{1 - 1.45z^{-1} + 0.624z^{-2}}$$

This expression can be implemented easily using any of the techniques discussed in OSB Chapter 6.

Design procedure using bilinear transformation

Determining design parameters

Here we will convert from continuous-time to discrete-time form using the bilinear transform

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

This produces the nonlinear relationship between continuous-time frequency and discrete-time frequency

$$\Omega = \frac{2}{T}\tan(\omega/2)$$

Converting the critical frequencies ω_p and ω_s to their continuous-time counterparts produces

$$\Omega_p = \frac{2}{T}\tan(\omega_p/2) = 0.4802/T$$

$$\Omega_s = \frac{2}{T}\tan(\omega_s/2) = 1.226/T$$

We use again the design equation

$$N = \frac{1}{2}\frac{\log(k_2/k_1)}{\log(\Omega_s/\Omega_p)} = 2.4546$$

Note that although we still need to have 3 poles, the actual fractional value of N smaller because of the greater ratio of stopband to passband edge frequencies provided by the nonlinear frequency warping of the bilinear transform.

We will match the specs exactly in the passband, although there is no specific need to do so when we use the bilinear transform. The equation for the critical frequency then becomes

$$\Omega_c = \frac{\Omega_p}{k_1^{1/2N}} = \frac{0.4806}{T}$$

Prototype filter design

As before, the transfer function for the Butterworth filter is

$$H(s) = \frac{\Omega_c^3}{(s-\Omega_c e^{j\pi})(s-\Omega_c e^{j(\pi+\pi/3)})(s-\Omega_c e^{j(\pi-\pi/3)})}$$

Using the MATLAB routine `poly` and the value for Ω_c that we had obtained we find that the transfer function of the prototype filter is

$$H(s) = \frac{0.110/T^3}{s^3 + \frac{0.9612s^2}{T} + \frac{0.4620s}{T^2} + \frac{0.110}{T^3}}$$

Conversion to discrete-time form

We can now substitute directly the bilinear transform relationship $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$

$$H(s) = \frac{0.110/T^3}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + \frac{0.9612\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{T} + \frac{0.4620\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}{T^2} + \frac{0.110}{T^3}}$$

Note that the design parameter T appears in cubic form in the denominator of every term in numerator and denominator. Hence its specific value does not matter and we can cancel it from the design equation. The value of the T parameter is frequently set to equal 2 to simplify the expression for the bilinear transform.

We also note that simplifying the expression above is an algebraic nightmare. Using the MATLAB routine `bilinear` with the poles and (null) zeros of $H(s)$ as input parameters (and a value of 1 for the parameter T), we obtain

$$H(z) = \frac{0.00854(1+3z^{-1}+3z^{-2}+z^{-3})}{1-2.064z^{-1}+1.519z^{-2}-0.386z^{-3}}$$