COMSATS Institute of Information Technology

EEE 324 Digital Signal Processing

Lecture 26

Discrete Fourier Transform

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Contents

- Discrete Fourier Transform

Discrete Fourier Transform

- Discrete Fourier Transform (DFT) corresponds to samples, equally spaced in frequency, of the Fourier transform of a finite-duration signal.
- Unlike Discrete Time Fourier Transform (DTFT), DFT is a sequence rather than a function of a continuous variable.
- DFT is important for the implementation of DSP algorithms since there exist efficient algorithms for its computation.
- The Fourier series representation of the periodic sequence corresponds to the DFT of the finite length sequence.

Discrete Fourier Transform

Analysis Equation

$$X[k] = \sum_{n=0}^{N-1} x[n] {W_N}^{kn}$$

Synthesis Equation

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] {W_N}^{-kn}$$

Where

$$W_N = e^{-j(\frac{2\pi}{N})}$$

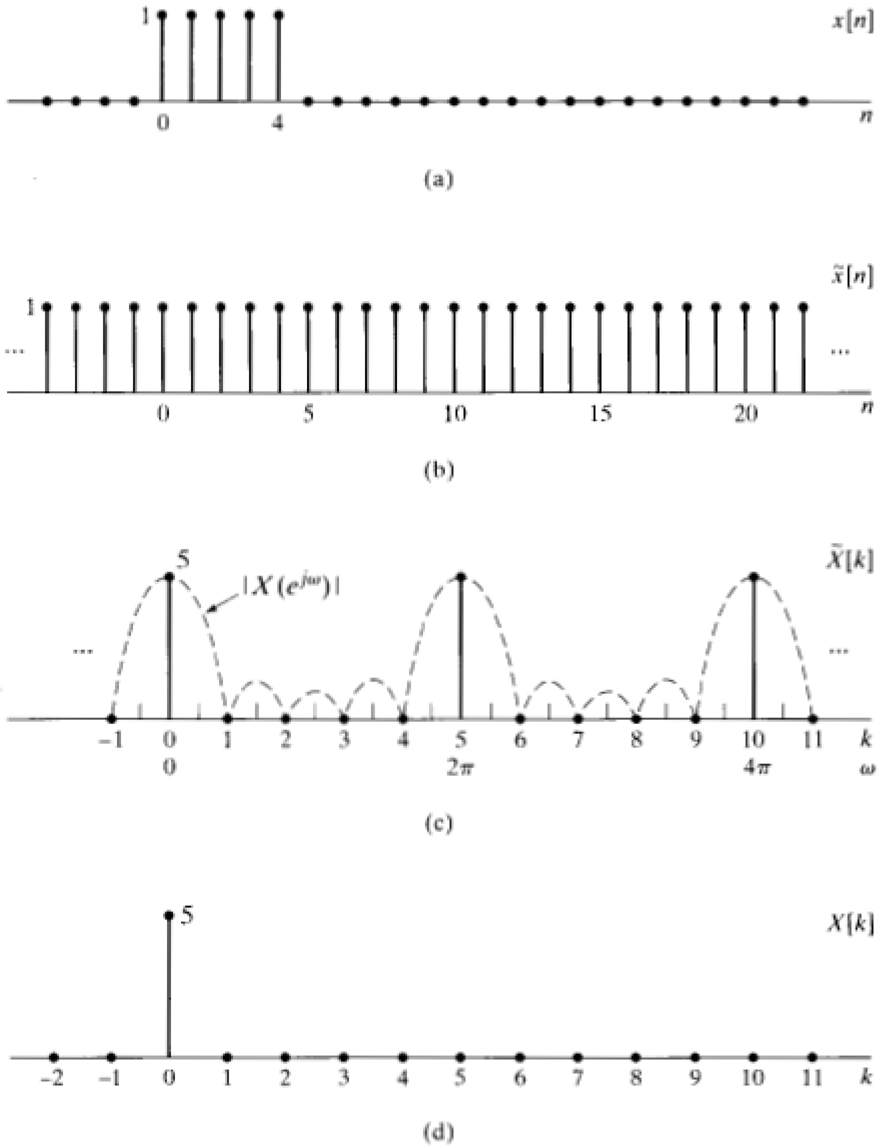
$$x[n] \overset{DFT}{\longleftrightarrow} X[k]$$

- The DFT $X[k]$ is equal to samples of the periodic Fourier transform $X(e^{j\omega})$

Discrete Fourier Transform

Example 8.7 (DFT of a Rectangular Pulse)

N = 5



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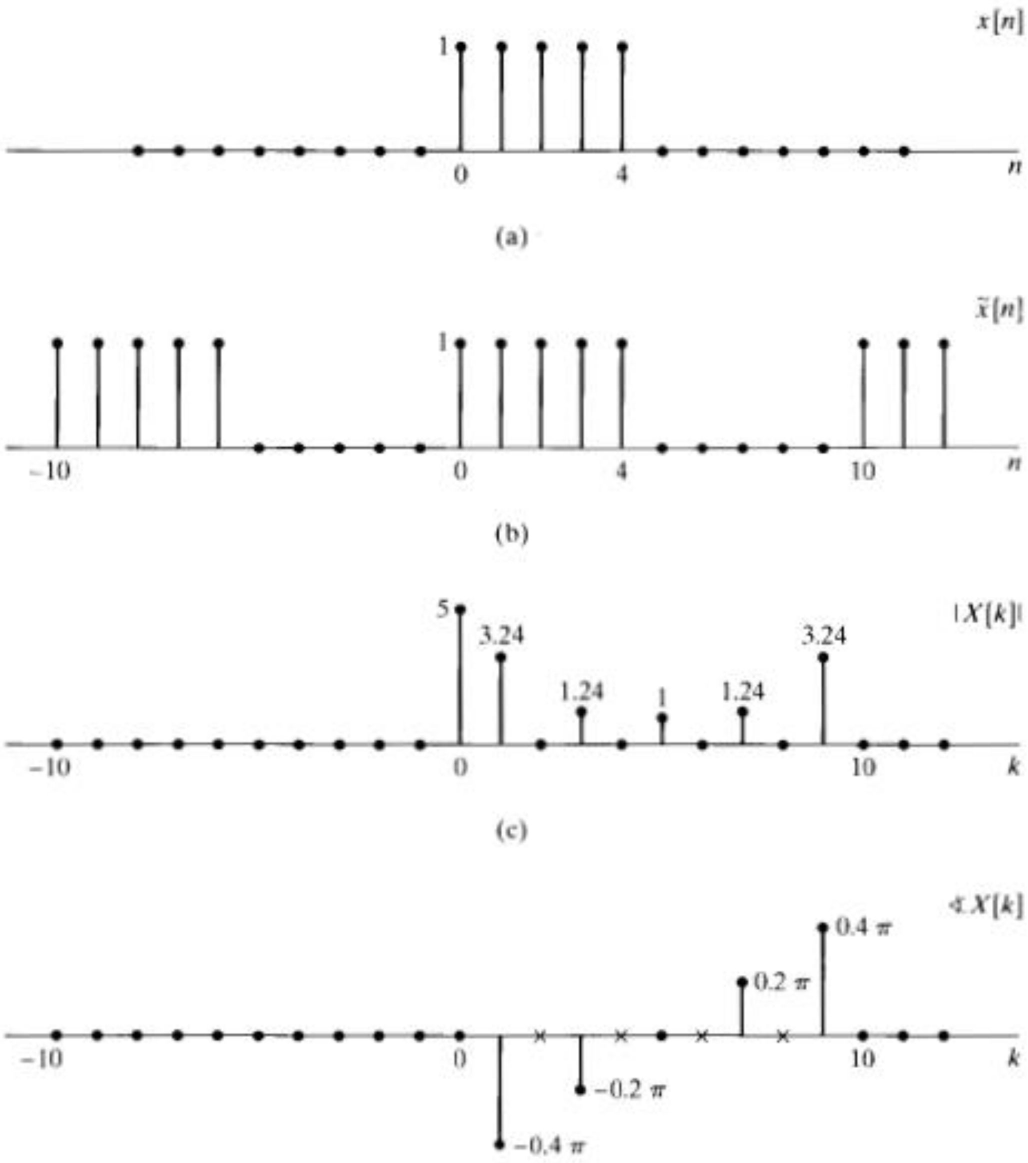
Discrete Fourier Transform

Example 8.7 (DFT of a Rectangular Pulse)

N = 10



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Discrete Fourier Transform

(Selected) Properties of DFT

1. Linearity
2. Circular Shift
3. Circular Convolution

Discrete Fourier Transform (Properties)

1. <u>**Linearity**</u>

- If two finite duration sequences $x_1[n]$ and $x_2[n]$ are linearly combined i.e., if

$$x_3[n] = ax_1[n] + bx_2[n]$$

then, the DFT of $x_3[n]$ is

$$X_3[k] = aX_1[k] + bX_2[k]$$

Discrete Fourier Transform (Properties)

1. Linearity

- If $x_1[n]$ has length N_1 and $x_2[n]$ has length N_2, then the maximum length of $x_3[n]$ will be $N_3 = \max[N_1, N_2]$.
- Hence, both DFTs must be computed with the same length $N \geq N_3$.
- If $N_1 < N_2$, then $X_1[k]$ must be augmented by $(N_2 - N_1)$ zeros.
- That is, the $N_2 - point$ DFT of $x_1[n]$ is

$$X_1[k] = \sum_{n=0}^{N_1-1} x_1[n] {W_{N_2}}^{kn} \ , \qquad 0 \leq \mathrm{k} \leq N_2 - 1$$

- And the $N_2 - point$ DFT of $x_2[n]$ is

$$X_2[k] = \sum_{n=0}^{N_2-1} x_1[n] {W_{N_2}}^{kn} \ , \qquad 0 \leq \mathrm{k} \leq N_2 - 1$$

Discrete Fourier Transform (Properties)

1. **Linearity**

In summary, if

$$x_1[n] \overset{DFT}{\longleftrightarrow} X_1[k]$$

And

$$x_2[n] \overset{DFT}{\longleftrightarrow} X_2[k]$$

Then

$$ax_1[n] + bx_2[n] \overset{DFT}{\longleftrightarrow} aX_1[k] + bX_2[k]$$

Discrete Fourier Transform (Properties)

2. <u>**Circular Shift of a Sequence**</u>

If

$$x_1[n] = x[\left((n-m)\right)_N]$$

Then

$$X_1[k] = {W_N}^{mk} X[k]$$

Discrete Fourier Transform (Properties)

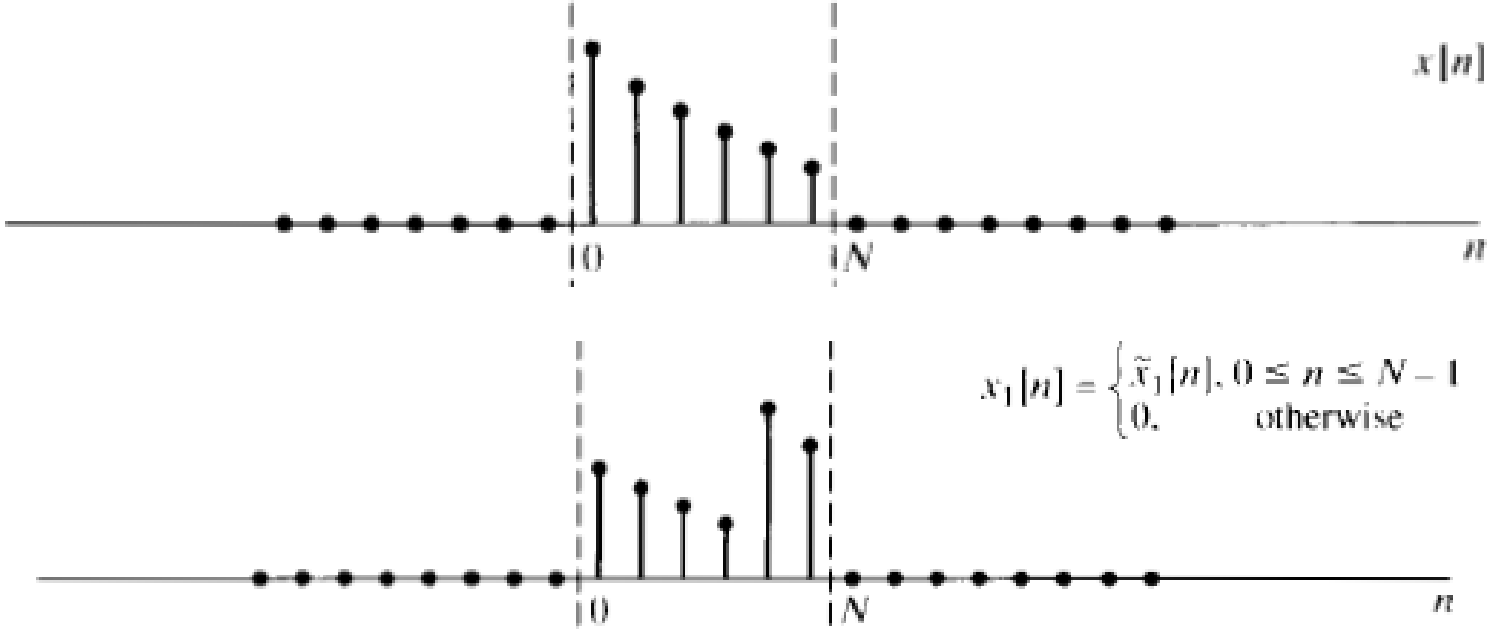
2. Circular Shift of a Sequence

Example

$m = -2$

$N = 6$

$x_1[n] = x[\left((n+2)\right)_6]$



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Discrete Fourier Transform (Properties)

2. **<u>Circular Shift of a Sequence</u>**

<u>Example 8.8</u>

If $x_2[n] = x[((n-4))_6]$

Then $x_2[n] = x_1[n]$

Because, generally,

$$W_N{}^{mk} = W_N{}^{-(N-m)k}$$

- That is, an $N - point$ circular shift in one direction by m is the same as a circular shift in the opposite direction by $N - m$

Discrete Fourier Transform (Properties)

3. **Circular Convolution**

$$x_3[n] = x_1[n] \, \textcircled{N} \, x_2[n]$$

$$x_3[n] = \sum_{m=0}^{N-1} x_1[((m))_N] x_2[((n-m))_N], \qquad 0 \le n \le N-1$$

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N], \qquad 0 \le n \le N-1$$

$$x_3[n] \overset{DFT}{\longleftrightarrow} X_1[k] X_2[k]$$

Discrete Fourier Transform (Properties)

3. **<u>Circular Convolution</u>**

<u>Example 8.11</u>

Case 1: N = L (Circular Convolution)

Case 2: N = 2L (Linear Convolution)

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1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$
11.	$\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12.	$j\mathcal{J}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.		
15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16.	$x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17.	$x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

Practice Problems

Problems: 8.5, 8.6, 8.10, 8.11, 8.13, 8.14.