EEE 324 Digital Signal Processing

Lecture 2

Sampling and Reconstruction of Sinusoidal Signals

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Contents

- Sampling and Reconstruction of a Sinusoidal Signal

Introduction

- In this lecture, with the help of three examples, we will see the effect of changing the sampling frequency on the sampling and reconstruction of sinusoidal signals.

Sampling and Reconstruction of a Sinusoidal Signal

<u>Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)</u>

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

To do:

1. Sample the above signal (Obtain $x[n]$)
2. Reconstruct the above signal from its samples obtained in Part 1 (Obtain $x_r(t)$ from $x[n]$ or $x_s(t)$)

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

$$x[n] = x_c(nT) = \cos(4000\pi nT) = \cos\left(\frac{4000\pi n}{6000}\right) = \cos\left(\frac{2\pi}{3}n\right)$$

Comparing with the general form of a sinusoid ($\cos(\omega_0 n)$)

$$\omega_0 = \frac{2\pi}{3}$$

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Sampling and Reconstruction of a Sinusoidal Signal

<u>Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)</u>

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

1. **<u>Sampling</u>**

a. Check for aliasing:

$$\Omega_s \geq 2\Omega_0?$$

$$\Omega_s = \frac{2\pi}{T} = 12000\pi$$

$$\Omega_0 = \frac{\omega_0}{T} = 4000\pi$$

$\Rightarrow$ No Aliasing (Nyquist conditions satisfied)

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

b. Find FT of $x_c(t)$

$$X_c(j\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

Recall that

$\cos \omega_0 t$ $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ $a_1 = a_{-1} = \frac{1}{2}$, $a_k = 0$, otherwise

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

c. Find FT of $x_s(t)$

We know that $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

For $\Omega_s = 12000\pi$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k12000\pi))$$

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

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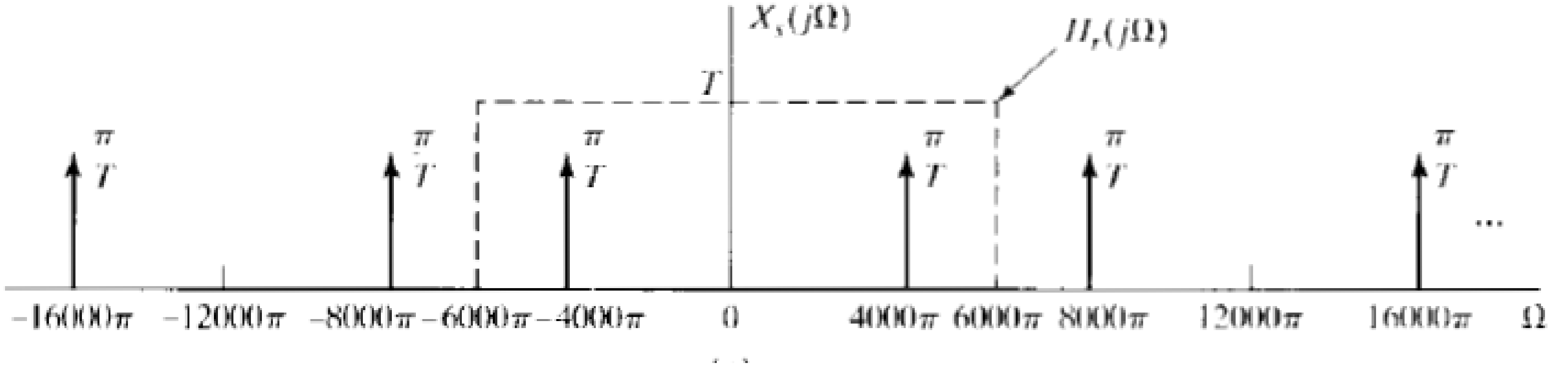

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)

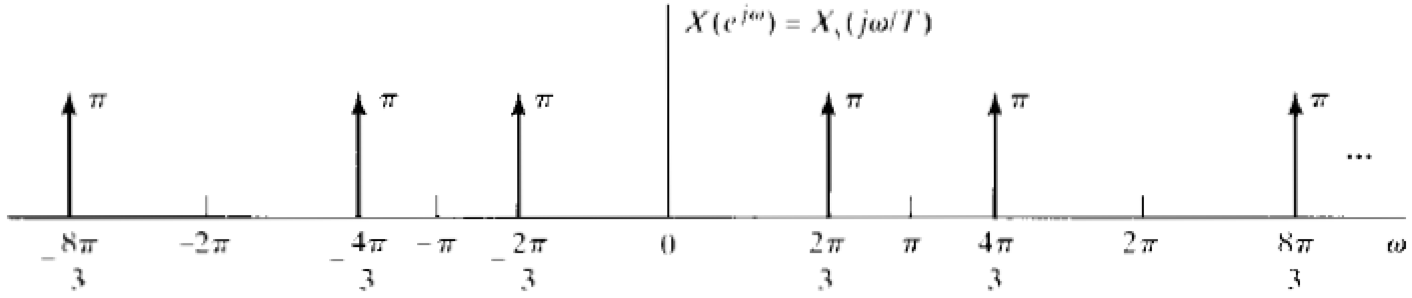
$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

d. Find FT of $x[n]$

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T}$$



Scaling the independent variable of an impulse also scales its area

i.e., $\delta\left(\frac{\omega}{T}\right) = T\delta(\omega)$

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)

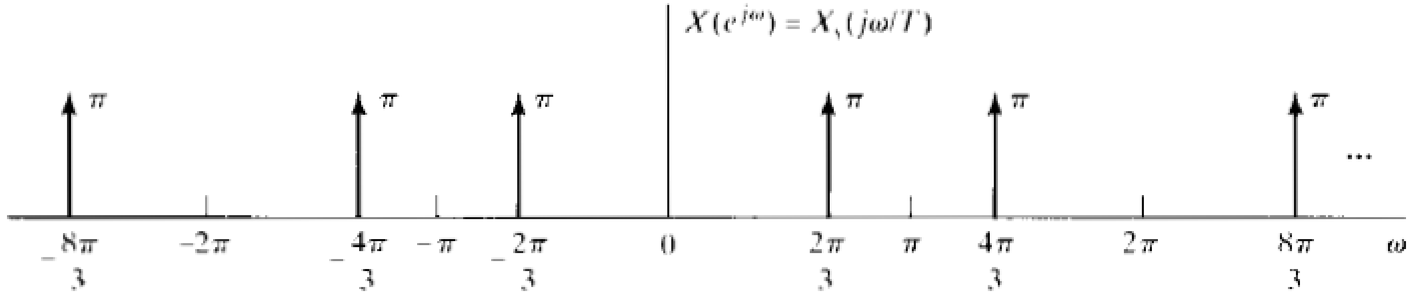
$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

e. Find $x[n]$

$$x[n] = \cos\left(\frac{2\pi}{3}n\right)$$



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Sampling and Reconstruction of a Sinusoidal Signal

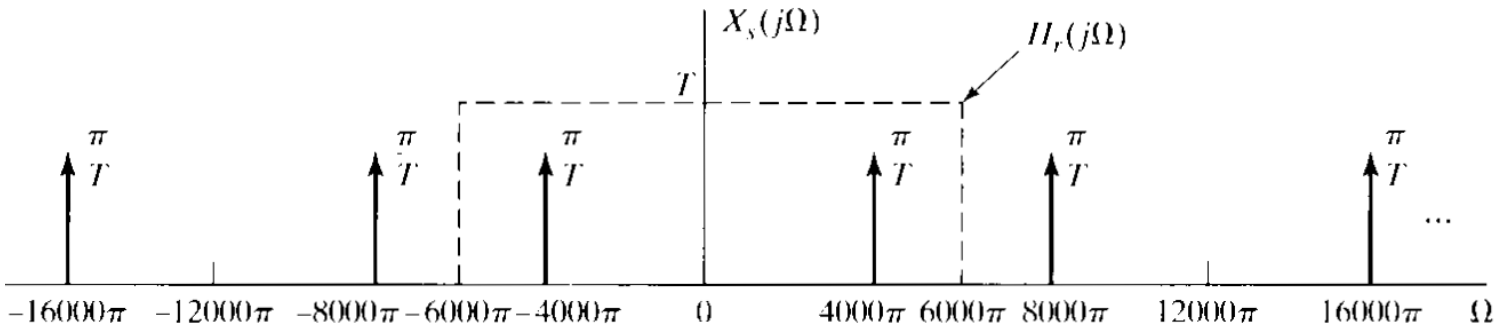
Example 4.1 (Sampling and Reconstruction of a Sinusoidal Signal)

$$x_c(t) = \cos(4000\pi t)$$
$$T = \frac{1}{6000}$$

1. **Reconstruction**

f. Find $x_r(t)$

$X_r(j\Omega) = X_s(j\Omega)H_r(j\Omega)$



$X_r(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$
$x_r(t) = \cos(4000\pi t)$
$\Rightarrow x_r(t) = x_c(t)$

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.2 (Aliasing in the Reconstruction of an Undersampled Sinusoidal Signal)

$$x_c(t) = \cos(16000\pi t)$$

$$T = \frac{1}{6000}$$

To do:

1. Sample the above signal (Obtain $x[n]$)
2. Reconstruct the above signal from its samples obtained in Part 1 (Obtain $x_r(t)$ from $x[n]$ or $x_s(t)$)

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.2 (Aliasing in the Reconstruction of an Undersampled Sinusoidal Signal)

$$x_c(t) = \cos(16000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

$$x[n] = x_c(nT) = \cos(16000\pi nT) = \cos\left(\frac{16000\pi n}{6000}\right) = \cos\left(\frac{8\pi}{3}n\right) = \cos\left(\frac{2\pi}{3}n\right)$$

Comparing with the general form of a sinusoid ($\cos(\omega_0 n)$)

$$\omega_0 = \frac{8\pi}{3}$$

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.2 (Aliasing in the Reconstruction of an Undersampled Sinusoidal Signal)

$$x_c(t) = \cos(16000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

a. Check for aliasing:

$$\Omega_s \geq 2\Omega_0?$$

$$\Omega_s = \frac{2\pi}{T} = 12000\pi$$

$$\Omega_0 = \frac{\omega_0}{T} = 16000\pi$$

$\Rightarrow$ Aliasing (Nyquist conditions not satisfied)

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.2 (Aliasing in the Reconstruction of an Undersampled Sinusoidal Signal)

$$x_c(t) = \cos(16000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

b. Find FT of $x_c(t)$

$$X_c(j\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

Recall that $\cos \omega_0 t$ $\quad$ $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ $\quad$ $a_1 = a_{-1} = \frac{1}{2}$, $a_k = 0$, otherwise

$$X_c(j\Omega) = \pi\delta(\Omega - 16000\pi) + \pi\delta(\Omega + 16000\pi)$$

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.2 (Aliasing in the Reconstruction of an Undersampled Sinusoidal Signal)

$$x_c(t) = \cos(16000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

c. Find FT of $x_s(t)$

We know that $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

For $\Omega_s = 12000\pi$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k12000\pi))$$

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.2 (Aliasing in the Reconstruction of an Undersampled Sinusoidal Signal)

$$x_c(t) = \cos(16000\pi t)$$

$$T = \frac{1}{6000}$$

1. **Sampling**

d. Find FT of $x[n]$

$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T}$

Scaling the independent variable of an impulse also scales its area

i.e., $\delta\left(\frac{\omega}{T}\right) = T\delta(\omega)$

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<u>Example 4.2 (Aliasing in the Reconstruction of an Undersampled Sinusoidal Signal)</u>

$$x_c(t) = \cos(16000\pi t)$$

$$T = \frac{1}{6000}$$

1. **<u>Sampling</u>**

e. Find $x[n]$

$$x[n] = \cos\left(\frac{2\pi}{3} n\right)$$

Sampling and Reconstruction of a Sinusoidal Signal

- We have obtained the same sequence of samples $x[n] = \cos\left(\frac{2\pi}{3}n\right)$ by sampling two different CT signals with the same sampling frequency.
- In one case, the sampling frequency satisfied the Nyquist criterion.
- In the other, the sampling frequency did not satisfy the Nyquist criterion.
- In Example 4.2, the reconstructed signal would have frequency $\Omega_0 = 4000\pi$, which is not the frequency of the original signal $x_c(t)$ in Example 4.2
- $x_r(t) = \cos(4000\pi t)$

$\Rightarrow x_r(t) \neq x_c(t)$

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.3 (A Second Example of Aliasing)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{1500}$$

To do:

1. Sample the above signal (Obtain $x[n]$)
2. Reconstruct the above signal from its samples obtained in Part 1 (Obtain $x_r(t)$ from $x[n]$ or $x_s(t)$)

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.3 (A Second Example of Aliasing)

$$x_c(t) = \cos(4000\pi t)$$

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1. **Sampling**

$$x[n] = x_c(nT) = \cos(4000\pi nT) = \cos\left(\frac{4000\pi n}{1500}\right) = \cos\left(\frac{8\pi}{3}n\right)$$

Comparing with the general form of a sinusoid ($\cos(\omega_0 n)$)

$$\omega_0 = \frac{8\pi}{3}$$

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.3 (A Second Example of Aliasing)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{1500}$$

1. **Sampling**

a. Check for aliasing:

$$\Omega_s \geq 2\Omega_0?$$

$$\Omega_s = \frac{2\pi}{T} = 3000\pi$$

$$\Omega_0 = \frac{\omega_0}{T} = 4000\pi$$

$\Rightarrow$ Aliasing (Nyquist conditions not satisfied)

Sampling and Reconstruction of a Sinusoidal Signal

Example 4.3 (A Second Example of Aliasing)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{1500}$$

1. <u>Sampling</u>

b. Find FT of $x_c(t)$

$$X_c(j\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

Recall that

$\cos \omega_0 t$ $\quad$ $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ $\quad$ $a_1 = a_{-1} = \frac{1}{2}$, $a_k = 0$, otherwise

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$$

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$$T = \frac{1}{1500}$$

1. **Sampling**

c. Find FT of $x_s(t)$

We know that $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

For $\Omega_s = 3000\pi$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k3000\pi))$$

Sampling and Reconstruction of a Sinusoidal Signal

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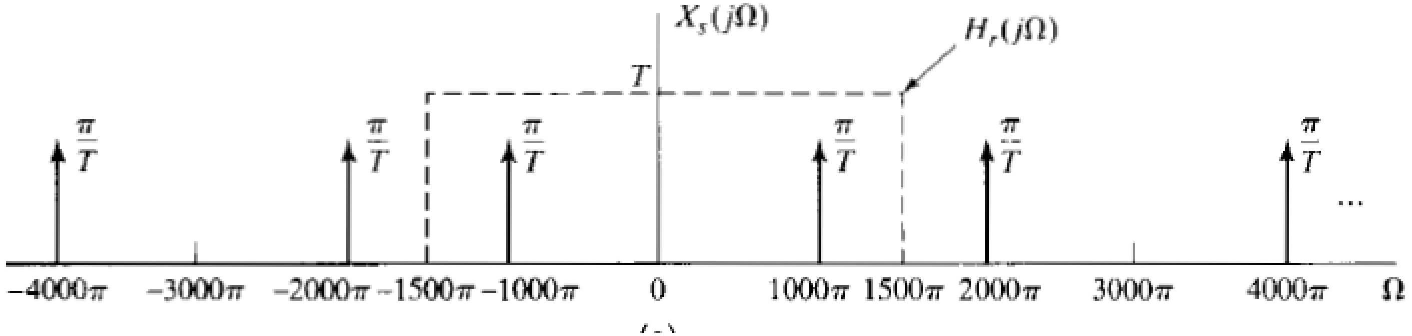

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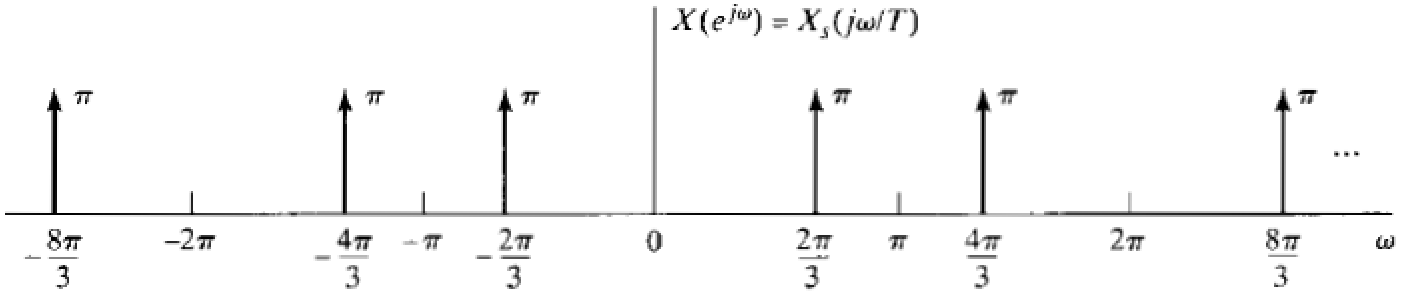
1. **<u>Sampling</u>**

d. Find FT of $x[n]$

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T}$$



Scaling the independent variable of an impulse also scales its area

i.e., $\delta\left(\frac{\omega}{T}\right) = T\delta(\omega)$

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$$x[n] = \cos\left(\frac{2\pi}{3}n\right)$$

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Sampling and Reconstruction of a Sinusoidal Signal

Example 4.3 (A Second Example of Aliasing)

$$x_c(t) = \cos(4000\pi t)$$

$$T = \frac{1}{1500}$$

- The same DT signal may result from sampling the same CT signal at two different sampling rates.
- The reconstructed CT signal would have frequency $\Omega_0 = 1000\pi$ and not 4000π.
- $x_r(t) = \cos(1000\pi t)$

$\Rightarrow x_r(t) \neq x_c(t)$

Take Home!

- It is possible to get the same DT sequence in the following two cases:
  - When two different CT signals are sampled using the same sampling rate.
  - When the same CT signal is sampled using two different sampling rates.
- Aliasing occurs when Nyquist condition is not satisfied during sampling.

Take Home!

- It is possible to get the same DT sequence in the following two cases:
  - When two different CT signals are sampled using the same sampling rate.
  - When the same CT signal is sampled using two different sampling rates.
- Aliasing occurs when Nyquist condition is not satisfied during sampling.

Reading

- Section 4.0 – 4.2 (Oppenheim)

Practice Problems

- Problems 4.1 – 4.4, 4.8 – 4.11 (Oppenheim)