EEE 324 Digital Signal Processing

Lecture 4

Down-sampling

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Contents

- Down-sampling

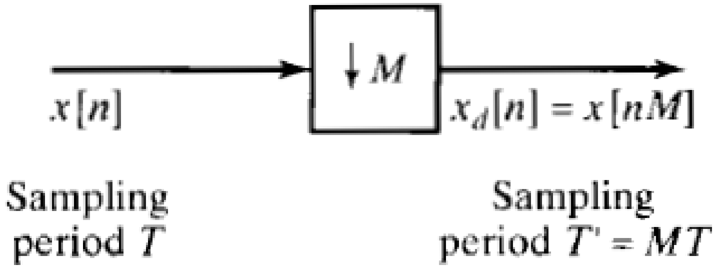
Down-Sampling (Reducing the Sampling Rate)

- The sampling rate of a sequence can be reduced by defining a new sequence

$$\boxed{x_d[n] = x[nM] = x_c(nMT)} \qquad (34)$$

Where M is a positive integer

- The system defined by Eq. (34) is called a ***compressor*** and is shown below.



- $x_d[n]$ is identical to the sequence that would be obtained from $x_c(t)$ by sampling with period $T' = MT$

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Down-Sampling (Reducing the Sampling Rate)

- Moreover, if $X_c(j\Omega) = 0 \; for \; |\Omega| \geq \Omega_N$, i.e. $x_c(t)$ is a BL signal,
  - then $x_d[n]$ is an exact representation of $x_c(t)$ if $\pi/_{T'} = \pi/_{MT} \geq \Omega_N$.
- The sampling rate can be reduced by a factor M without aliasing using two techniques: i.e.,
  1. if: the original sampling rate was at least M times the Nyquist rate (i.e., $\Omega_s \geq M2\Omega_0$), or
  2. if the bandwidth of the sequence is first reduced by a factor of M by DT filtering.

Down-Sampling (Reducing the Sampling Rate)

Frequency Domain Relationship Between I/P and O/P of Compressor

- Since the compressor is a DT system, we will employ DTFT.
- The DTFT of $x\,[n] = x_c(nT)$ is:

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$ (Recall Eq. (10))

- The DTFT of $x_d[n] = x[nM] = x_c(nT')$ with $T' = MT$ is:

$$X_d\left(e^{j\omega}\right) = \frac{1}{T'}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'} - \frac{2\pi r}{T'}\right)\right) \quad (35)$$ (Recall Eq. (10))

Putting the value of T',

$$X_d\left(e^{j\omega}\right) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right) \quad (36)$$

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Frequency Domain Relationship Between I/P and O/P of Compressor

$$r = i + kM, \qquad -\infty < k < \infty, \qquad 0 \le i \le M - 1$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right)] \quad (37)$$

Down-Sampling (Reducing the Sampling Rate)

Frequency Domain Relationship Between I/P and O/P of Compressor

- The term inside the square brackets of Eq. (37) resemble Eq. (10) and can be written as:

$$X\left(e^{j(\omega-2\pi i)/M}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{(\omega-2\pi i)}{MT} - \frac{2\pi k}{T}\right)\right)$$

$Hence, Eq. (37) can\ be\ expressed\ as:$

$$X_d\left(e^{j\omega}\right) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j(\omega-2\pi i)/M}\right) \qquad (38)$$

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<u>Frequency Domain Relationship Between I/P and O/P of Compressor</u>

From Eq. (36), we can say that, in frequency domain, down-sampling by a factor of M can be thought of as being composed of:

- An *infinite* set of *copies* of $X_c(j\Omega)$, frequency *scaled* through $\omega = \Omega T'$ and,
- *Shifted* by integer multiples of $\frac{2\pi}{T'}$

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Frequency Domain Relationship Between I/P and O/P of Compressor

From Eq. (38), we can say that:

- in frequency domain, down-sampling by a factor of M produces ***M aliased copies*** of the DTFT $X(e^{j\omega})$.
- These copies are produced by:
  - ***Stretching*** the frequency axis by a factor of M,
  - and then ***shifting*** by $2\pi i$.

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Frequency Domain Relationship Between I/P and O/P of Compressor

Either interpretation makes it clear that:

- $X_d(e^{j\omega})$ is periodic with period 2π, and

Down-Sampling (Reducing the Sampling Rate)

Frequency Domain Relationship Between I/P and O/P of Compressor

Two ways of avoiding aliasing during downsampling

1. Aliasing can be avoided by ensuring that $X(e^{j\omega})$ is band-limited i.e.,

$$X\left(e^{j\omega}\right) = 0, \qquad \omega_N \leq |\omega| \leq \pi$$

And

$$\frac{2\pi}{M} \geq 2\omega_N$$

Or

$$M\omega_N \leq \pi$$

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Frequency Domain Relationship Between I/P and O/P of Compressor

$$\Omega_s = 4\Omega_N, M = 2$$

$$\Omega_N = \frac{1}{4}\Omega_s$$

$$\omega_N = \frac{\pi}{2}$$

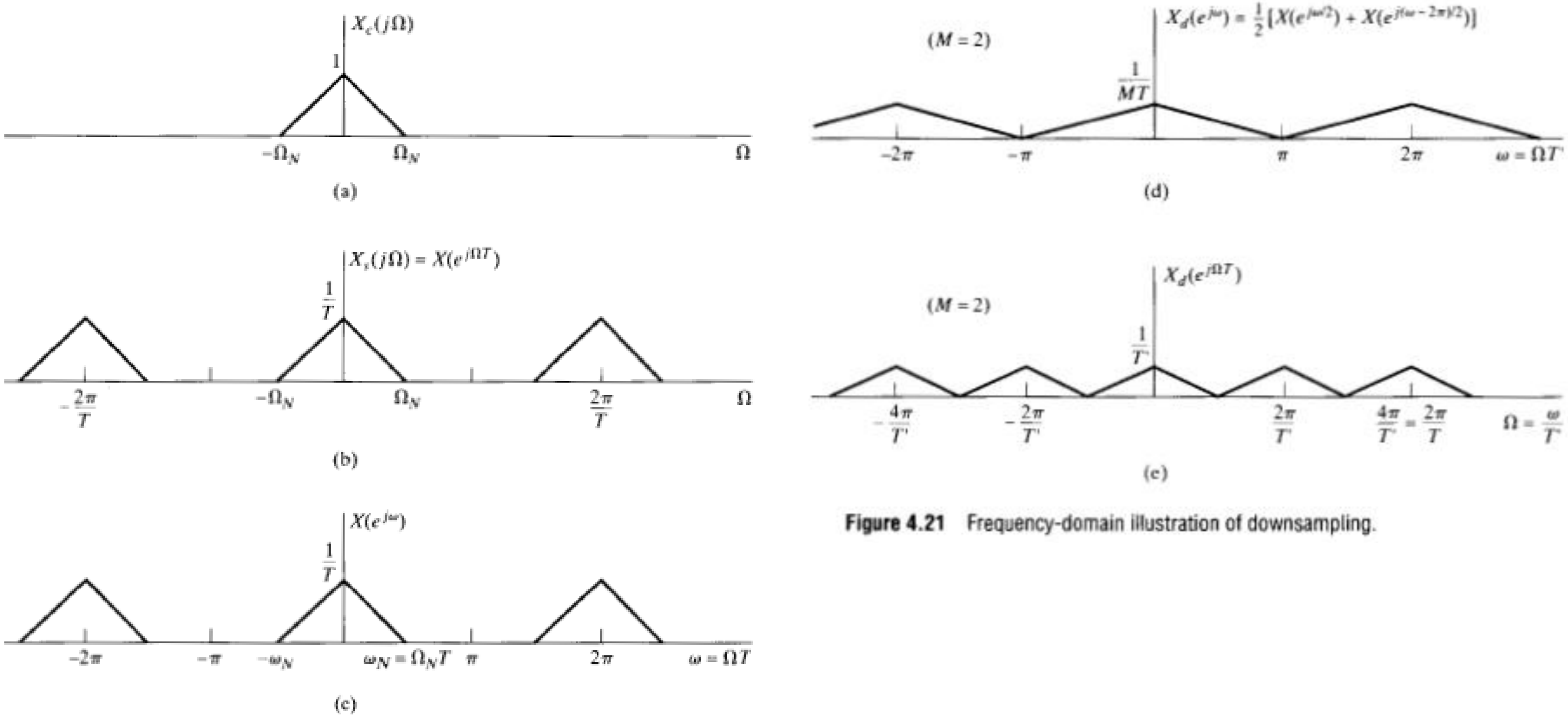

Figure 4.21 Frequency-domain illustration of downsampling.

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Frequency Domain Relationship Between I/P and O/P of Compressor

$\Omega_s = 4\Omega_N$, $M = 3$

$$\Omega_N = \frac{1}{4}\Omega_s$$

$$\omega_N = \frac{\pi}{2}$$

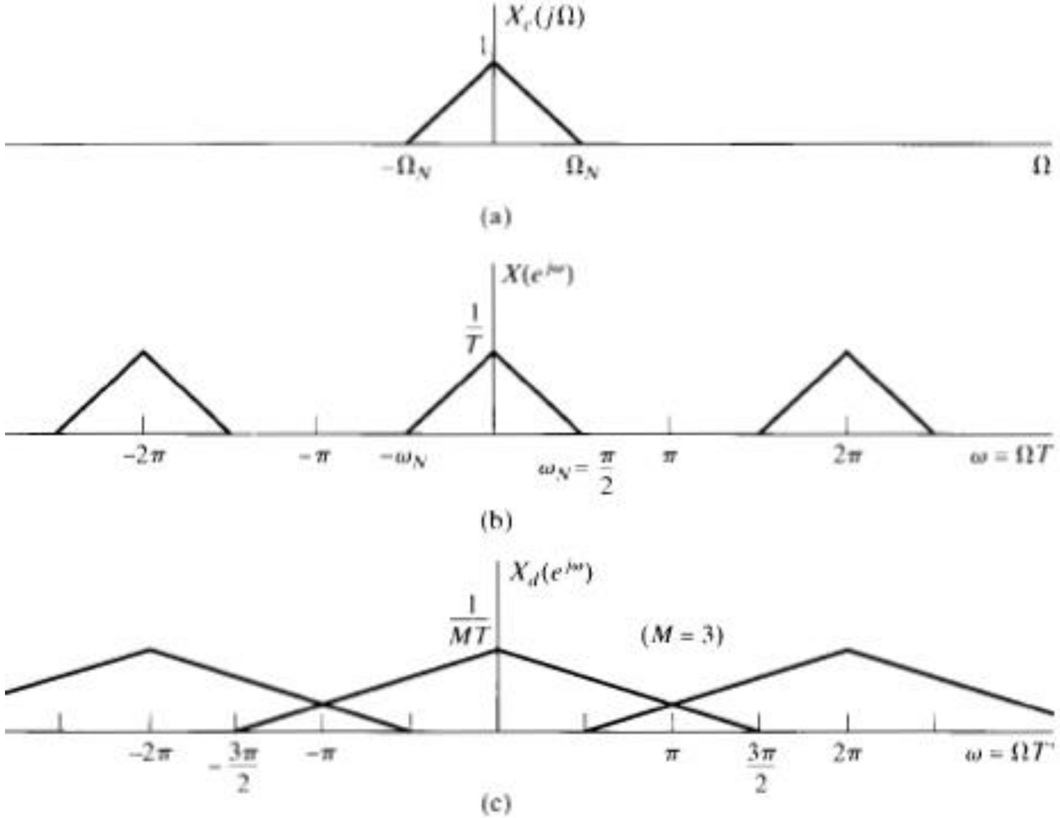

Figure 4.22 (a)–(c) Downsampling with aliasing.

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Frequency Domain Relationship Between I/P and O/P of Compressor

Two ways of avoiding aliasing during downsampling

2. Reduce the bandwidth of the signal $x[n]$ first by using an LP filter with cut-off frequency π/M and then down-sample.

For the example shown on the right, the following parameters are used:

$$\Omega_s = 4\Omega_N, M = 2$$

$$\Omega_N = \frac{1}{4}\Omega_s$$

$$\omega_N = \frac{\pi}{2}$$

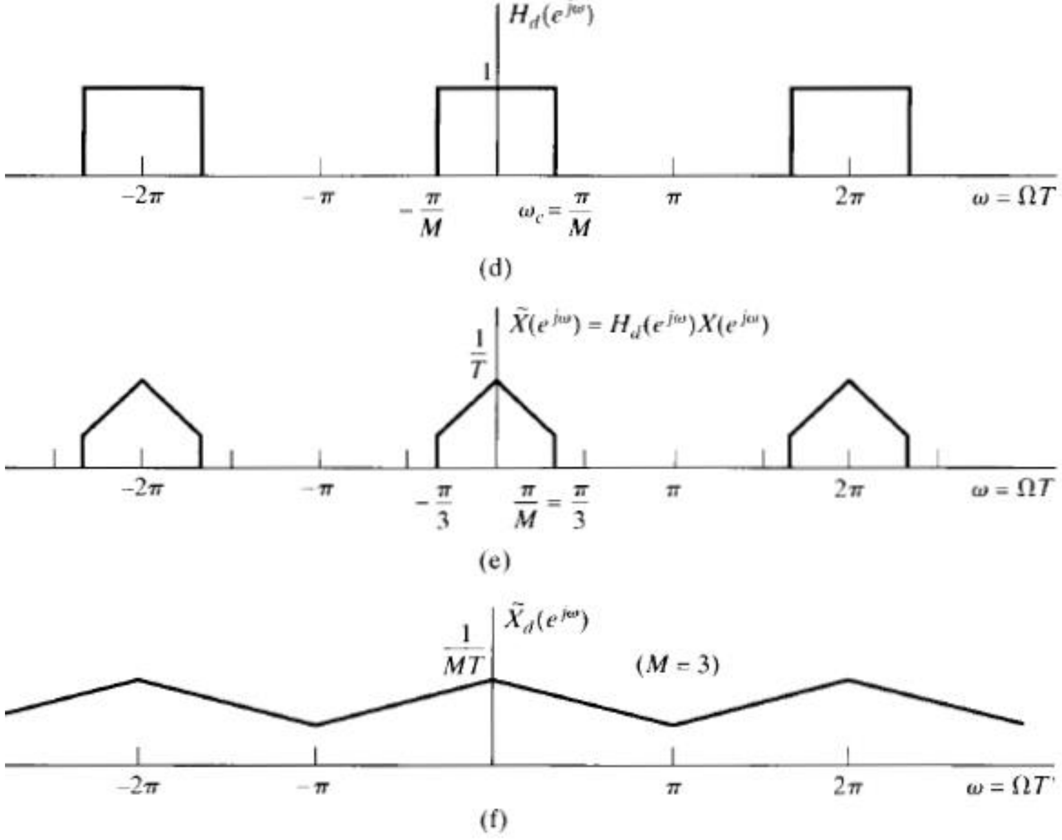

Figure 4.22 (a)–(c) Downsampling with aliasing. (d)–(f) Downsampling with prefiltering to avoid aliasing.

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Frequency Domain Relationship Between I/P and O/P of Compressor

- If the original signal was band-limited, such that:

$$X\left(e^{j\omega}\right) = 0, \quad \frac{\pi}{M} \leq |\omega| \leq \pi$$

  there would be no aliasing.

- To avoid aliasing, we need to first low-pass filter, and then down-sample.

$$H_d\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| < \pi/M \\ 0, & \dfrac{\pi}{M} \leq |\omega| \leq \pi \end{cases}$$

- In practice, due to unfavourable properties of $h_d[n]$ (infinite impulse response, slow decay), we do not apply this idealized filter.

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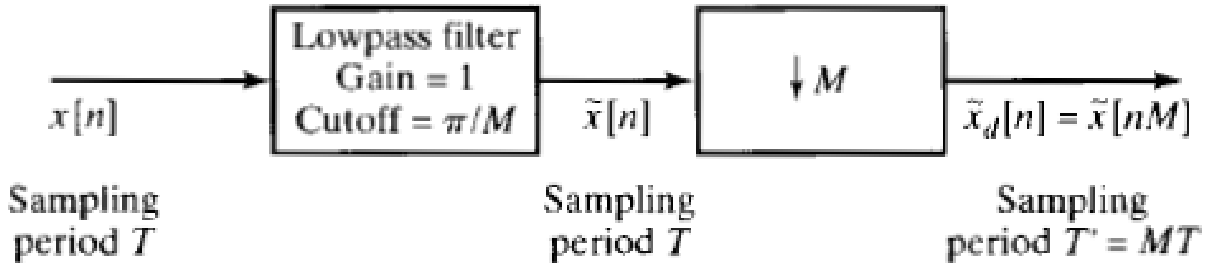
Frequency Domain Relationship Between I/P and O/P of Compressor

- A general system for down-sampling by a factor of M is shown below.



- Such a system is called a ***decimator***, and
- down-sampling by low-pass filtering followed by compression is called ***decimation.***

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Reading

<u>Section 4.6.1 (Oppenheim)</u>

Practice Problems

Problems: 4.14 – 4.18, 4.26, 4.27 (Oppenheim)

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