EEE 324 Digital Signal Processing

Lecture 5

Up-sampling

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Contents

- Up-sampling

Up-Sampling (Increasing the Sampling Rate)

- The objective is to obtain samples $x_i[n] = x_c(nT')$ from the sequence of samples $x[n] = x_c(nT)$ where $T' = T/L$ where L is a positive integer.
- The operation of increasing the sampling rate is called ***upsampling***.

$$x_i[n] = x\left[\frac{n}{L}\right] = x_c\left(n\frac{T}{L}\right), \qquad \text{n} = 0, \pm L, \pm 2L \ldots$$

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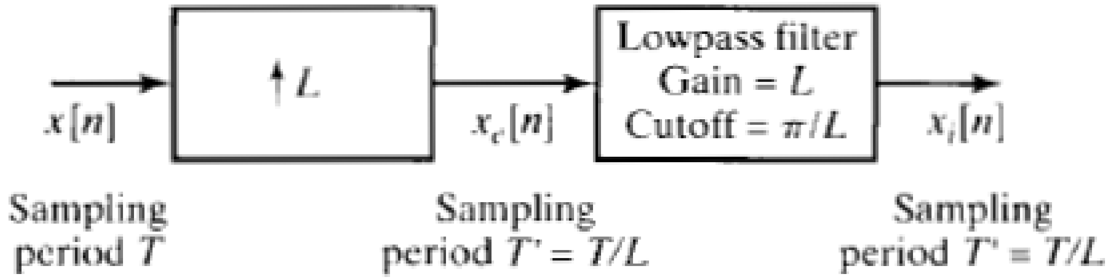
Up-Sampling (Increasing the Sampling Rate)

Frequency Domain Representation of Up-Sampling

- A general system for up-sampling by a factor of L is shown below.



- The system on the left: Expander (or Sampling rate expander)

$$x_e[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = kL, \quad k \in Z \\ 0, & else \end{cases}$$

Equivalently,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL] \qquad (38a)$$

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Frequency Domain Representation of Up-Sampling

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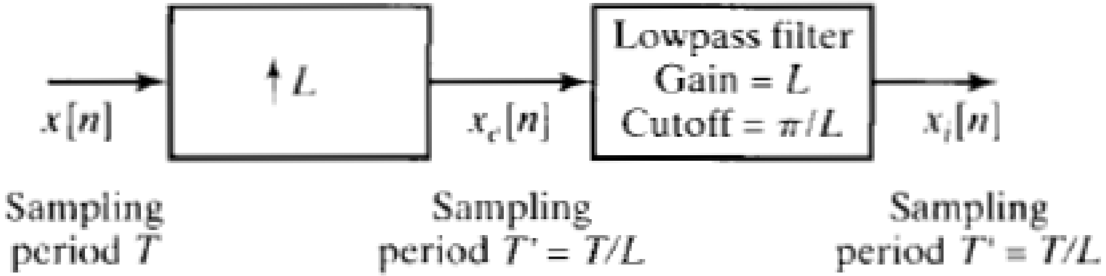

- The system on the right: DT LPF with COF = π/L and gain L
- Such a system is called an ***interpolator***, and
- Up-sampling is therefore considered as ***interpolation***.

Up-Sampling (Increasing the Sampling Rate)

Frequency Domain Representation of Up-Sampling

- Taking the DTFT of Eq. (38a),

$$X_e\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right)e^{j\omega n} \quad (39)$$

$$X_e\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right) \quad (40)$$

- Up-sampling contracts the DTFT by a factor of L.
- This can cause aliasing in high frequencies.
- To remove aliasing, we can low-pass filter $x_e[n]$, by applying an ideal low-pass filter

$$H_e\left(e^{j\omega}\right) = \begin{cases} L, & |\omega| < \pi/L \\ 0, & \frac{\pi}{L} \le |\omega| \le \pi \end{cases}$$

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Up-Sampling (Increasing the Sampling Rate)

<u>Frequency Domain Representation of Up-Sampling</u>

$$\Omega_s = 2\Omega_N$$

$$\Omega_N = \frac{1}{2}\Omega_s$$

$$\omega_N = \pi$$

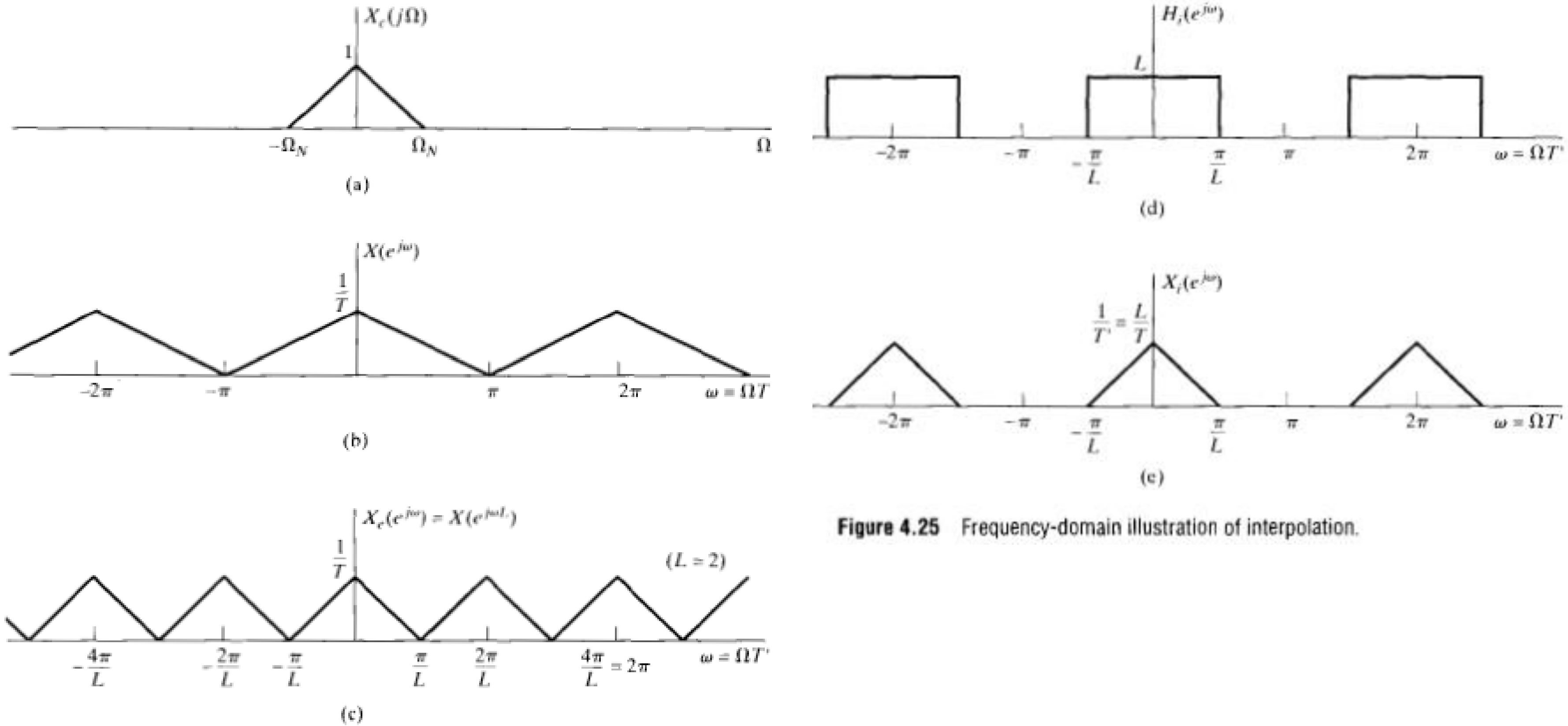

Figure 4.25 Frequency-domain illustration of interpolation.

Up-Sampling (Increasing the Sampling Rate)

<u>Frequency Domain Representation of Up-Sampling</u>

Interpolation Formula for $x_i[n]$ in terms of $x[n]$

$$h_i[n] = \frac{\sin(\frac{\pi n}{L})}{\frac{\pi n}{L}}$$

$$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wn}{\pi}\right)$$

$$0 < W < \pi$$

$$X(\omega) = \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}$$

$X(\omega)$ periodic with period 2π

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin(\frac{\pi(n-kL)}{L})}{\frac{\pi(n-kL)}{L}}$$

$$h_1[0] = 1$$

$$h_i[n] = 0, \qquad n = \pm L, \pm 2L, \ldots$$

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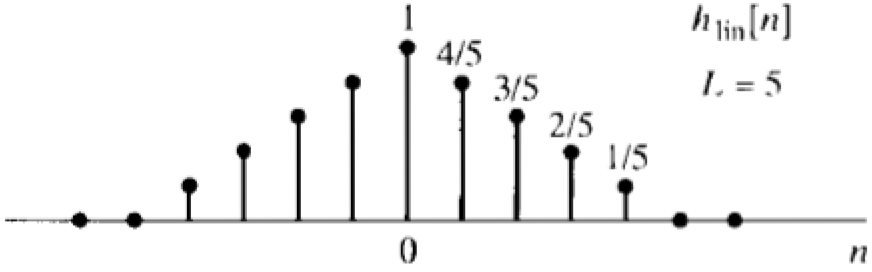
Up-Sampling (Increasing the Sampling Rate)

<u>Frequency Domain Representation of Up-Sampling</u>

Linear Interpolation

$$h_{lin}[n] = \begin{cases} 1 - \frac{|n|}{L}, |n| \leq L \\ 0, \quad\quad otherwise \end{cases}$$

For $L = 5$



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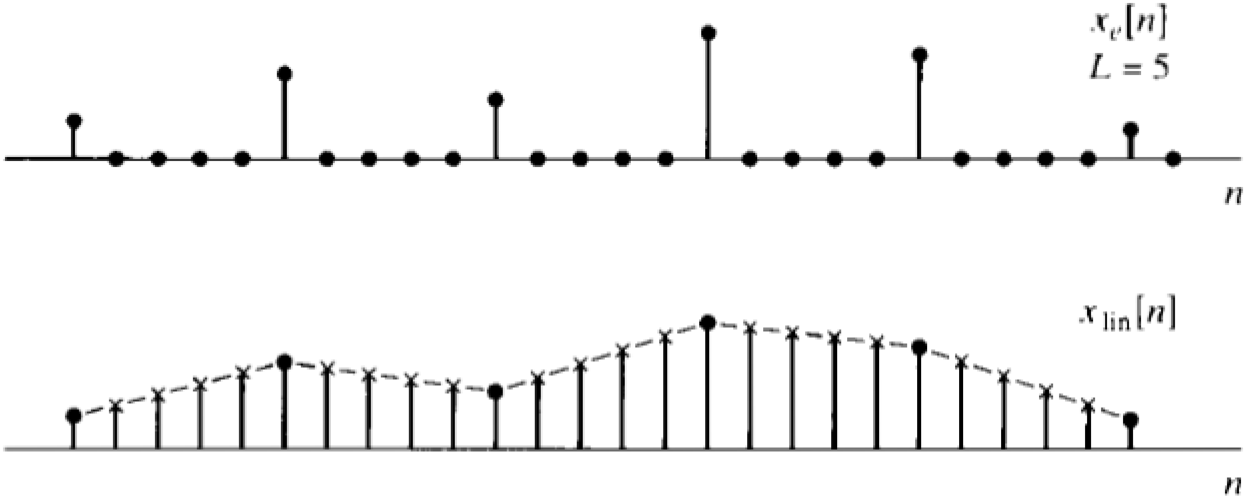
Up-Sampling (Increasing the Sampling Rate)

Frequency Domain Representation of Up-Sampling

Linear Interpolation

$$x_{lin}[n] = \sum_{k=-\infty}^{\infty} x_e[k]h_{lin}[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h_{lin}[n-kL]$$



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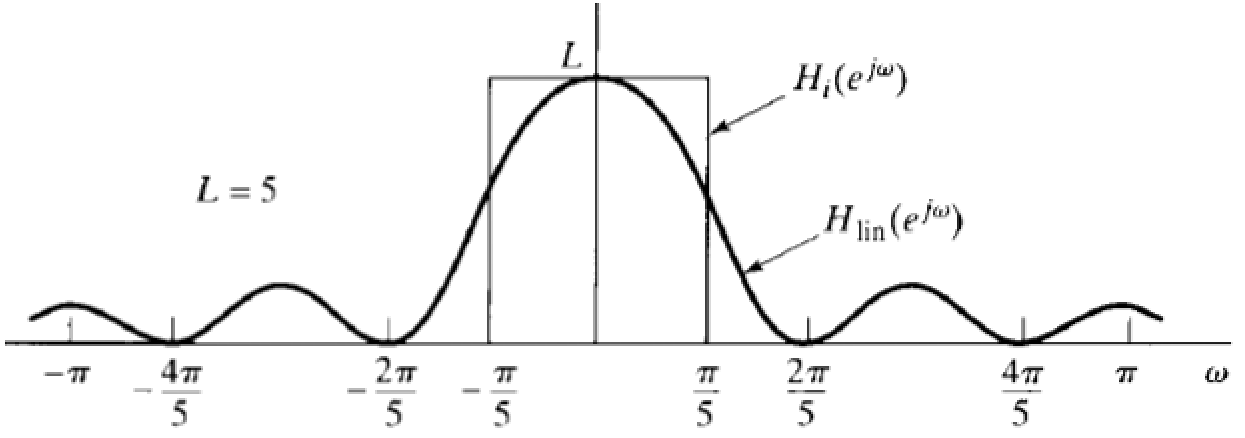
Up-Sampling (Increasing the Sampling Rate)

<u>Frequency Domain Representation of Up-Sampling</u>

Linear Interpolation

$$H_{lin}\left[e^{j\omega}\right] = \frac{1}{L}\left[\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right]^2$$



Up-Sampling (Increasing the Sampling Rate)

Frequency Domain Representation of Up-Sampling

Linear Interpolation

- If the original signal is sampled at the Nyquist rate, linear interpolation will not be very good, since the O/P of the filter will contain considerable energy in the band $\frac{\pi}{L} < |\omega| \leq \pi$.
- If the original sampling rate is much higher than Nyquist rate, then the LI will be more successful in removing the frequency-scaled images of $X_c(j\Omega)$ at multiples of $2\pi/L$.