EEE 324 Digital Signal Processing

Lecture 7

DT Processing of CT Signals

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Contents

- DT Processing of CT Signals

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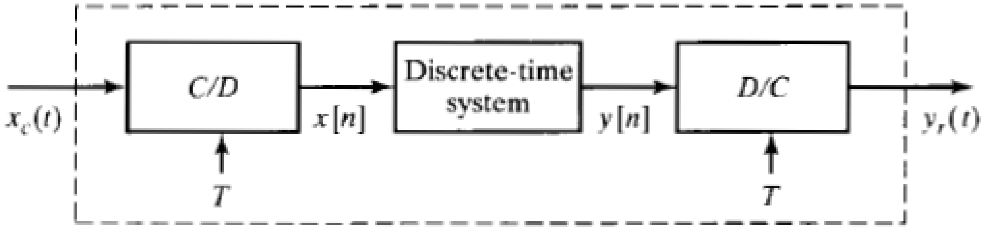
DT Processing of CT Signals

- DT Systems are commonly used to process CT signals.
- This is generally achieved by a system shown below.



- The overall system is equivalent to a CT system since it has a CT input $x_c(t)$ and a CT output $y_r(t)$.
- The properties of the overall system depend on the choice of the sampling rate $1/T$ and the DT system.
- Although not essential, we assume the same T for C/D and D/C.
- To avoid aliasing, $x_c(t)$ should be band-limited i.e., $\Omega_N \leq \frac{\Omega_s}{2}$.
    - If that is not the case, an anti-aliasing filter can be used before the C/D converter.

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Summarizing the C/D converter

- The C/D converter takes a CT input $x_c(t)$and produces a DT signal $x[n]$ i.e.,

$$x[n] = x_c(nT)$$

- The DTFT of the output signal is related to the CTFT of the input signal i.e.,

$$\boxed{X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)} \quad (15)$$

DT Processing of CT Signals

Summarizing the D/C converter

- The D/C converter takes a DT input $y[n]$ and reconstructs a CT signal $y_r(t)$ i.e.,

$$\boxed{y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin\left[\pi\left(\frac{t-nT}{T}\right)\right]}{\frac{\pi(t-nT)}{T}}} \quad (16) \quad \text{(Recall Eq. (13), Lecture 3)}$$

- The CTFT of the output signal is related to the DTFT of the input signal i.e.,

$$\boxed{Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})} \quad (17) \quad \text{(Recall Eq. (14), Lecture 3)}$$

$$\boxed{Y_r(j\Omega) = \begin{cases} TY\left(e^{j\Omega T}\right), & |\Omega| < \frac{\pi}{T} \\ 0, & otherwise \end{cases}} \quad (17a)$$

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<u>Relating the O/P sequence $\boldsymbol{y[n]}$ to the I/P sequence $\boldsymbol{x[n]}$</u>

- Or equivalently, $Y(e^{j\omega})$ to $X(e^{j\omega})$
- Let us consider that the DT system is LTI.
- We know that for an LTID system,

$$\boxed{Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)} \qquad (18)$$

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The behaviour of the cascade system

- Combining Eq. (17) and Eq. (18),

$$\boxed{Y_r(j\Omega) = H_r(j\Omega)H\left(e^{j\Omega T}\right)X\left(e^{j\Omega T}\right)} \qquad (19)$$

- Also using Eq. (15) with $\omega = \Omega T$,

$$\boxed{Y_r(j\Omega) = H_r(j\Omega)H\left(e^{j\Omega T}\right)\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)} \qquad (20)$$

- If $X_c(j\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$, then the ideal reconstruction filter $H_r(j\Omega)$ cancels the factor $\frac{1}{T}$ and selects only the term in Eq. (20) for $k = 0$; i.e.,

$$\boxed{Y_r(j\Omega) = \begin{cases} H\left(e^{j\Omega T}\right)X_c(j\Omega), & |\Omega| < \frac{\pi}{T} \\ 0, & otherwise \end{cases}} \qquad (21)$$

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The behaviour of the cascade system

- If $X_c(j\Omega)$ is band-limited and the sampling rate is above the Nyquist rate, the output is related to the input through an equation of the form

$$\boxed{Y_r(j\Omega) = H_{eff}(j\Omega)X_c(j\Omega)} \qquad (22)$$

Here,

$$\boxed{H_{eff}(j\Omega) = \begin{cases} H\left(e^{j\Omega T}\right), & |\Omega| < \frac{\pi}{T} \\ 0, & otherwise \end{cases}} \qquad (23)$$

- So the overall CT system is equivalent to an LTIC system whose effective frequency response is given in Eq. (23).

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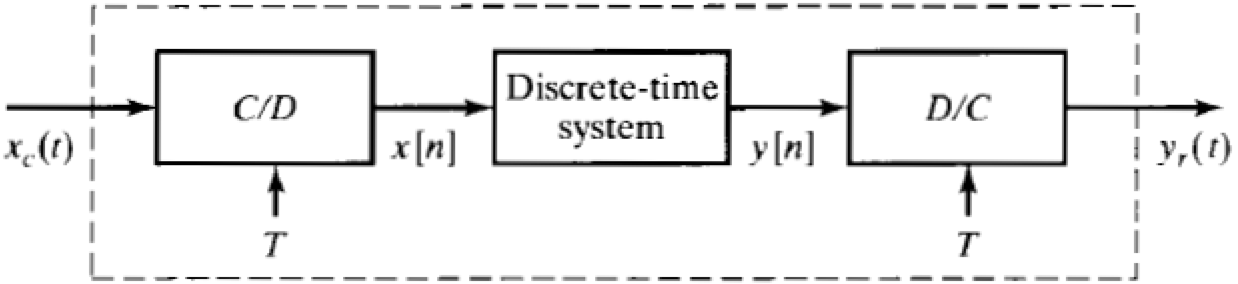
<u>The behaviour of the cascade system</u>

<u>Example 4.4 (Ideal CT LP Filtering using a DT LP Filter)</u>

Consider the following cascade system.



Let the LTI DT System have the frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

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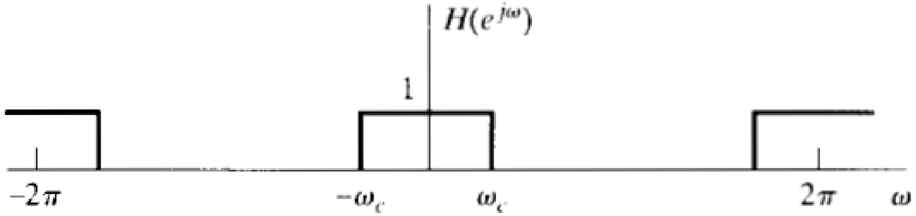
The behaviour of the cascade system

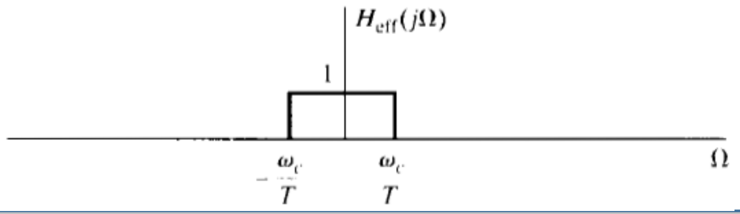
Example 4.4 (Ideal CT LP Filtering using a DT LP Filter)

- As it is a DT system, the FR is periodic with period 2π



- For a BL I/P sampled above the Nyquist rate, according to Eq. (23), the overall system will behave as an LTI CT system with FR

$$H_{eff}(j\Omega) = \begin{cases} 1, & |\Omega T| < \omega_c \; or \; |\Omega| < \dfrac{\omega_c}{T} \\ 0, & \omega_c < |\Omega T| \; or \; \dfrac{\omega_c}{T} < |\Omega| \end{cases}$$



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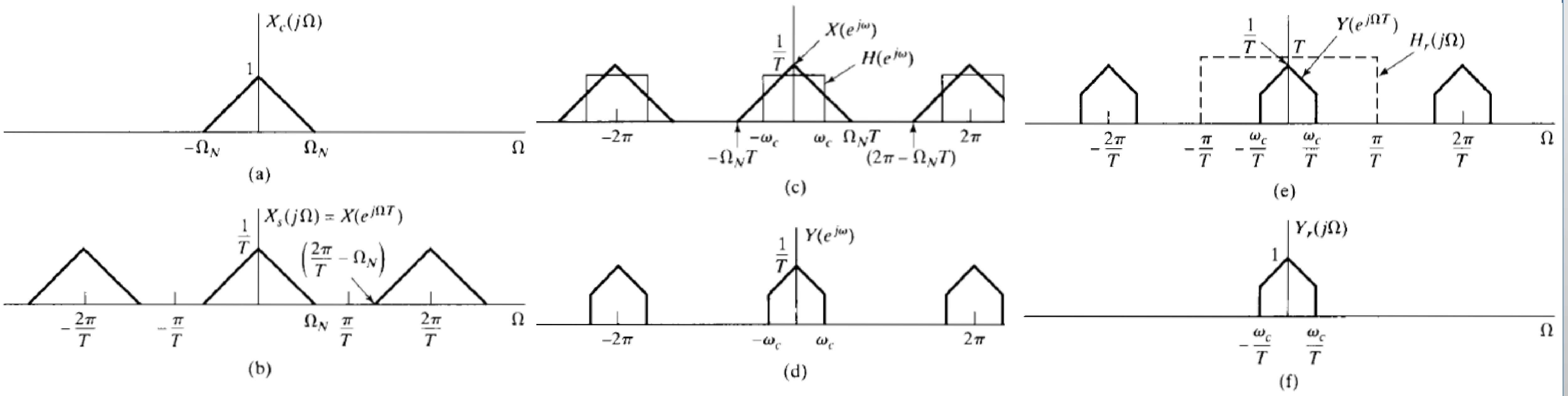
The behaviour of the cascade system

Example 4.4 (Ideal CT LP Filtering using a DT LP Filter)



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<u>The behaviour of the cascade system</u>

<u>Example 4.4 (Ideal CT LP Filtering using a DT LP Filter)</u>

- The Ideal DT LPF with DT cut-off freq. ω_c has the effect of an ideal LPF with cut-off freq. $\Omega_c = \frac{\omega_c}{T}$ when used in the cascade configuration used in this lecture.
- This cut-off freq. depends both on ω_c and T.
- By using a fixed DT LPF, but varying the sampling period T, an equivalent CT LPF with a variable cut-off freq. can be implemented.
- E.g., if T is chosen such that $\Omega_N T < \omega_c$, then the O/P would be $y_r(t) = x_c(t)$

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The behaviour of the cascade system

Example 4.4 (Ideal CT LP Filtering using a DT LP Filter)

- To avoid aliasing, it is required that:

$$(2\pi - \Omega_N T) > \omega_c$$

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<u>The behaviour of the cascade system</u>

<u>Example 4.5 (DT Implementation of an Ideal CT BL Differentiator)</u>

The ideal CT differentiator is defined by

$$y_c(t) = \frac{d}{dt}[x_c(t)]$$

with FR

$$H_c(j\Omega) = j\Omega$$

For the CT cascade system with BL input, the effective FR $H_{eff}(j\Omega)$ is

$$H_{eff}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \frac{\pi}{T} \\ 0, & otherwise \end{cases}$$

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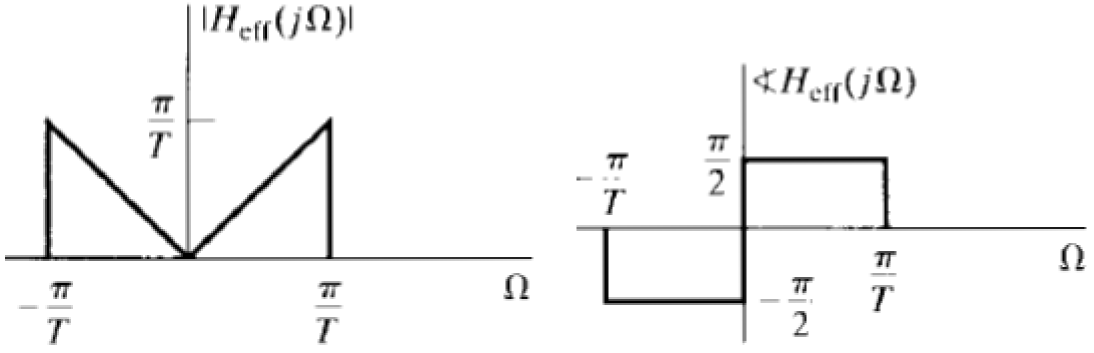
Example 4.5 (DT Implementation of an Ideal CT BL Differentiator)



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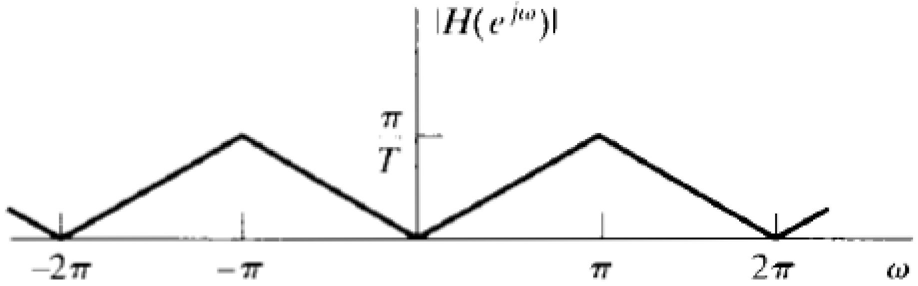
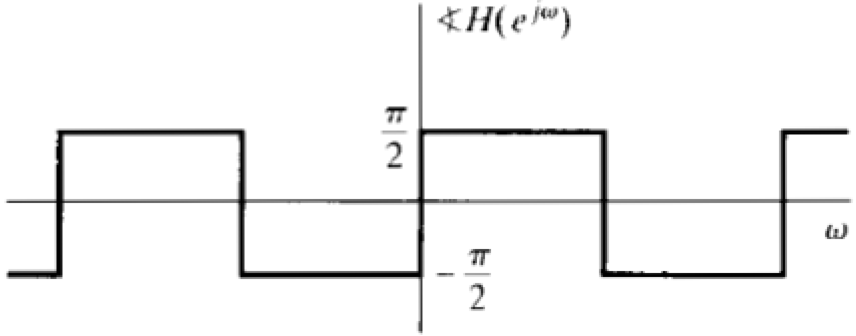
<u>The behaviour of the cascade system</u>

<u>Example 4.5 (DT Implementation of an Ideal CT BL Differentiator)</u>

To obtain the corresponding DT system, use $\omega = \Omega T$. So,

$$H\left(e^{j\omega}\right) = \frac{j\omega}{T}, \qquad |\omega| < \pi$$





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<u>The behaviour of the cascade system</u>

<u>Example 4.5 (DT Implementation of an Ideal CT BL Differentiator)</u>

The IR of the DT system is

$$h[n] = \frac{\pi n cos\pi n - sin\pi n}{\pi n^2 T}, \qquad -\infty < n < \infty$$

Or

$$h[n] = \begin{cases} 0, & n = 0 \\ \dfrac{cos\pi n}{nT}, & n \neq 0 \end{cases}$$

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The behaviour of the cascade system

Example 4.6 (Illustration of Ex. 4.5 with Sinusoidal I/P)

Let the I/P in Ex. 4.5 be

$$x_c(t) = \cos(\Omega_0 t) \text{ with } \Omega_0 < \frac{\pi}{T}$$

$$x[n] = \cos(\omega_0 n), \qquad \omega_0 = \Omega_0 T < \pi$$

DTFT in terms of Ω,

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty}[\pi\delta(\Omega - \Omega_0 - k\Omega_s) + \pi\delta(\Omega + \Omega_0 - k\Omega_s)]$$

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Example 4.6 (Illustration of Ex. 4.5 with Sinusoidal I/P)

When we focus on the baseband freqs. $-\frac{\pi}{T} < \Omega < \frac{\pi}{T}$, we obtain

$$X\left(e^{j\Omega T}\right) = \frac{\pi}{T}\delta(\Omega - \Omega_0) + \frac{\pi}{T}\delta(\Omega + \Omega_0), \text{ for } |\Omega| \leq \frac{\pi}{T}$$

In terms of ω, using the property $\delta\left(\frac{\omega}{T}\right) = T\delta(\omega)$,

$$X\left(e^{j\omega}\right) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0), \text{ for } |\omega| \leq \pi$$

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Example 4.6 (Illustration of Ex. 4.5 with Sinusoidal I/P)

The DTFT of the output of the DT system is

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X(e^{j\omega})$$

$$Y\left(e^{j\omega}\right) = \frac{j\omega}{T}\left[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)\right]$$

$$Y\left(e^{j\omega}\right) = \frac{j\pi\omega_0}{T}\delta(\omega-\omega_0)-\frac{j\pi\omega_0}{T}\delta(\omega+\omega_0), \quad |\omega|\leq\pi$$

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<u>The behaviour of the cascade system</u>

<u>Example 4.6 (Illustration of Ex. 4.5 with Sinusoidal I/P)</u>

The CTFT of the output of the overall CT cascade System, for $|\Omega| \leq \frac{\pi}{T}$

$$Y_r(j\Omega) = H_r(j\Omega)Y\left(e^{j\Omega T}\right) = TY\left(e^{j\Omega T}\right)$$

$$= T\left[\frac{j\pi\omega_0}{T}\delta(\Omega T - \Omega_0 T) - \frac{j\pi\omega_0}{T}\delta(\Omega T + \Omega_0 T)\right]$$

$$= T\left[\frac{j\pi\omega_0}{T}\left(\frac{1}{T}\right)\delta(\Omega - \Omega_0) - \frac{j\pi\omega_0}{T}\left(\frac{1}{T}\right)\delta(\Omega + \Omega_0)\right]$$

$$= j\Omega_0\pi\delta(\Omega - \Omega_0) - j\Omega_0\pi\delta(\Omega + \Omega_0)$$

And the Inverse CTFT is

$$y_r(t) = j\Omega_0\left(\frac{1}{2}\right)e^{j\Omega_0 t} - j\Omega_0\left(\frac{1}{2}\right)e^{-j\Omega_0 t} = -\Omega_0\sin(\Omega_0 t)$$

Which is the derivative of the input.

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Impulse Invariance

- Suppose we have a system as shown to right.
- Let us demand that,

$$\boxed{H\left(e^{j\omega}\right) = H_c\left(\frac{j\omega}{T}\right),\ |\omega| < \pi} \qquad (24)$$

$$\boxed{H_c(j\Omega) = 0,\ |\Omega| \geq \frac{\pi}{T}} \qquad (25)$$

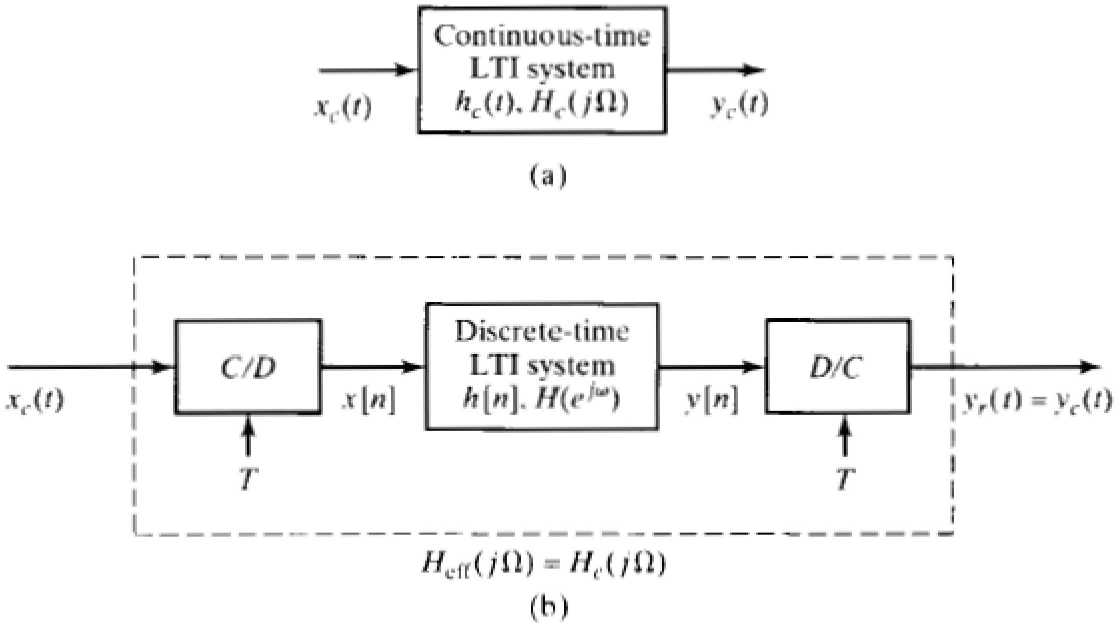

Figure 4.15 (a) Continuous-time LTI system. (b) Equivalent system for bandlimited inputs.

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Impulse Invariance

- When Eq. (24) and Eq. (25) hold,

$$\boxed{h[n] = Th_c(nT)} \qquad (26)$$

$$H(e^{j\omega}) = H_c\left(\frac{j\omega}{T}\right), \quad |\omega| \leq \pi \qquad (26a)$$

- The above equations hold for band-limited signals.
- Hence, the impulse response of the DT system is a scaled, sampled, version of $h_c(t)$.
- When Eq. (26) holds, the DT system is said to be the ***impulse invariant*** version of the CT system.

DT Processing of CT Signals

Impulse Invariance

Example 4.7 (DT LPF obtained using Impulse Invariance)

Given:

FR of a CT System: $H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & otherwise \end{cases}$

IR: $h_c(t) = \frac{\sin(\Omega_0 t)}{\pi t}$

Find:

IR of a DT System: $h[n]$

$$h[n] = T h_c(nT) = T\,\frac{\sin(\Omega_0 nT)}{\pi nT} = \frac{\sin(\omega_0 n)}{\pi n}$$

FR: $H\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$

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Impulse Invariance

Example 4.8 (Impulse Invariance Applied to CT Systems with Rational TFs)

Given:

IR: $h_c(t) = Ae^{s_0 t}u(t)$

SF: $H_c(s) = \frac{A}{s-s_0}$

To Do: Apply Impulse Invariance

$$h[n] = Th_c(nT) = Ae^{s_0 Tn}u[n]$$

TF: $H(z) = \frac{AT}{1-e^{s_0 T}z^{-1}}$

FR: $H\left(e^{j\omega}\right) = \frac{AT}{1-e^{s_0 T}e^{-j\omega}}$

DT Processing of CT Signals

Impulse Invariance

Example 4.8 (Impulse Invariance Applied to CT Systems with Rational TFs)

- Here Eq. (26a) does not hold exactly, as the original CT system did not have a strictly BL FR.
- Hence, resulting DT FR is an aliased version of $H_c(j\Omega)$.
- The effect of aliasing may be small.
- Hence, one approach to the DT simulation of CT system and to the design of Digital filters is through sampling of the IR of a corresponding analog filter.

Practice Problems

Problems: 4.5, 4.6, 4.12, 4.13, 4.19 – 4.28 (Oppenheim)