COMSATS Institute of Information Technology

EEE 324 Digital Signal Processing

Lecture 8

z-Transform

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- z-Transform
- Properties of ROC for z-transform.

Z-Transform

Introduction

- In this lecture, we develop the z-transform representation of a sequence and study how the properties of a sequence are related to the properties of its z-transform.
- The z-transform for DT signals is the counterpart of the Laplace transform for CT signals.
- Both z-transform and Laplace transform have a similar relationship to DTFT and CTFT respectively.
- Advantages of z-transform over DTFT,
  - DTFT does not converge for all sequences while z-transform is a generalization of DTFT which encompasses a broader class of signals.
  - In analytical problems, z-transform notation is often more convenient than DTFT notation.

Z-Transform

- The DTFT of a sequence $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Z-transform can be obtained by replacing $e^{j\omega}$ with z; i.e.,

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}} \quad (1)$$

- $Z\{.\}$ is the notation used for z-transform. i.e.,

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Another notation for z-transform is:

$$x[n] \overset{Z}{\leftrightarrow} X(z)$$

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Z-Transform

- The z-transform in Eq. (1) is called the ***bi-lateral or two-sided z-transform***.
- Another type of z-transform is the ***unilateral or one-sided z-transform*** which is defined as:

$$\boxed{X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}} \quad (2)$$

- The bilateral and unilateral z-transforms are equivalent if $x[n] = 0, n < 0$.
- In this course, we only focus on bilateral z-transform.

Z-Transform

Existence of z-transform

Sufficient condition:

- The series $x[n]z^{-n}$ is absolutely summable. i.e.,

$$\boxed{\sum_{n=-\infty}^{\infty}|x[n]z^{-n}| < \infty} \quad (3)$$

Z-Transform

Relationship between DTFT and z-transform

- z-transform reduces to DTFT, if we replace z with $e^{j\omega}$ in Eq. (1).
- In other words, when z has a unity magnitude $|z| = 1$, the transform corresponds to DTFT.

$$z = re^{j\omega}$$

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n}$$

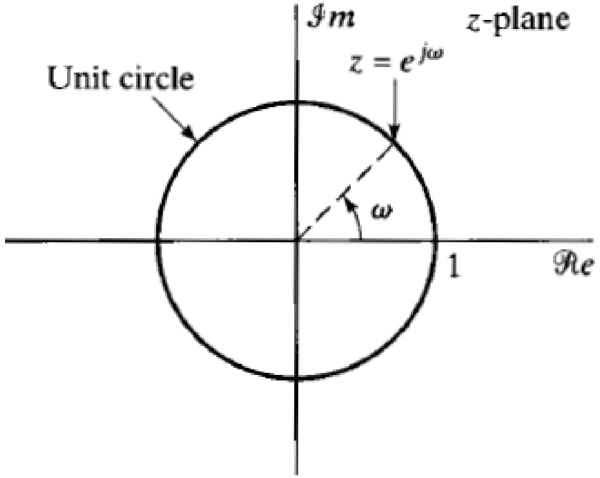
$$\boxed{X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right)e^{-j\omega n}} \quad (4)$$

- From Eq. (4), we can conclude that z-transform can be considered as the DTFT of $x[n]r^{-n}$.
- When $r = |z| = 1$, then, z-transform reduces to DTFT.

Z-Transform

The Unit Circle in the Complex z-plane

- In the z-plane, the contour corresponding to $|z| = 1$ is a circle of unit radius as shown below.



- This contour is referred to as the ***unit circle***.
- The z-transform evaluated on the unit circle corresponds to the DTFT.

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Z-Transform

Region of Convergence (ROC)

- The set of values of z for which the z-transform converges, is called the region of convergence, or simply, ROC.
- We know that Fourier transform converges if the sequence is absolutely summable.
- Since z-transform is the DTFT $x[n]r^{-n}$, the above condition can be modified for the convergence of z-transform i.e.. $\boxed{\sum_{n=-\infty}^{\infty}|x[n]r^{-n}| < \infty}$ (4)
- Because of the multiplication by the real exponential r^{-n}, it is possible for the z-transform to converge even if the DTFT does not.
- E.g., the sequence $x[n] = u[n]$ is not absolutely summable and hence, its DTFT does not exist. But $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform of the unit step exists with an ROC: $|z| > 1$.
- If some value $z = z_1$ is in the ROC, then all the values of z, on the circle defined by $|z| = |z_1|$ will also be in the ROC.

Z-Transform

Region of Convergence (ROC)

- Hence, the ROC consists of a ring in the z-plane centered about the origin.
- The outer boundary of the ring will be a circle (which may extend outward to infinity)
- The inner boundary of the ring will be a circle (which may extend inward to include the origin).

Z-Transform

Region of Convergence (ROC)

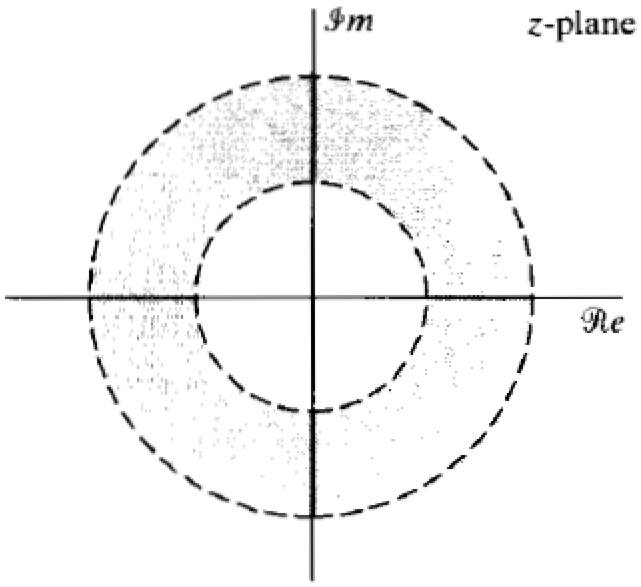

Figure 3.2 The region of convergence (ROC) as a ring in the *z*-plane. For specific cases, the inner boundary can extend inward to the origin, and the ROC becomes a disc. For other cases, the outer boundary can extend outward to infinity.

Z-Transform

Region of Convergence (ROC)

- If the ROC includes the unit circle,
  - Z-transform for $|z| = 1$, or equivalently the DTFT, converges
- If the ROC does not include the unit circle,
  - The DTFT does not converge.

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Z-Transform

Rational z-transform

- Among the most useful z-transforms are those for which $X(z)$ is a rational function inside the ROC, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}$$

Where $P(z)$ and $Q(z)$ are polynomials in z.

- The values of z for which $X(z) = 0$ are called the ***zeros*** of $X(z)$.
- The values of z for which $X(z) = \infty$ are called the ***poles*** of $X(z)$.
- In other words, zeros are the roots of the numerator while poles are the roots of the denominator.
- For rational z-transform, a number of important relationships exist between the locations of the poles of $X(z)$ and the ROC of the z-transform.
- $X(z)$ will always be rational when $x[n]$ is a linear combination of real or complex exponentials.
- For causal sequences, $X(z)$ involves only negative powers of z. Hence, it is convenient to express $X(z)$ in terms of z^{-1} rather than z.

Z-Transform

<u>Example 1: (Right sided exponential sequence)</u>

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Example 2: (Left sided exponential sequence)

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Example 3: (Sum of two exponential sequence)

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Example 4: (Sum of two exponential sequence)

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Example 5: (Two sided exponential sequence)

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Example 6: (Finite length sequence)

- The ROC of a finite length sequence is the entire z-plane.

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Common z-Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0\le n\le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Properties of ROC

1. The ROC is a ring or disk in the z-plane centered at the origin
2. The DTFT of $x[n]$ converges absolutely only if the ROC of the z-transform of $x[n]$ includes the unit circle.
3. The ROC cannot contain any poles.
4. The ROC of a finite duration sequence is the entire z-plane, except possibly $z = 0$ or $z = \infty$.
5. For a right-sided sequence, the ROC extends outward from the outermost finite pole in $X(z)$ to (and possibly including) $z = \infty$.
6. For a left-sided sequence, the ROC extends inward from the innermost (smallest magnitude) non-zero pole in $X(z)$ to (and possibly including) $z = 0$.
7. For a two-sided infinite duration sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, not containing any poles.
8. The ROC must be a connected region.

Properties of ROC

<u>Example 7: Stability, Causality and the ROC</u>

<u>Stability</u>

- $h[n]$ is absolutely summable
- $h[n]$ has a DTFT
- ROC must include the unit circle.

<u>Causality</u>

- $h[n]$ must be right sided.
- ROC is the region outward of the outermost pole.

Pole Location and Time Domain Behaviour of Causal Signals

$$x[n] = a^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - az^{-1}},$$

$$ROC: |z| > |a|$$

- The signal is decaying, if the pole is inside the unit circle.
- The signal is fixed, if the pole is on the unit circle.
- The signal is growing, if the pole is outside the unit circle.
- A negative pole results in a signal that alternates in sign.

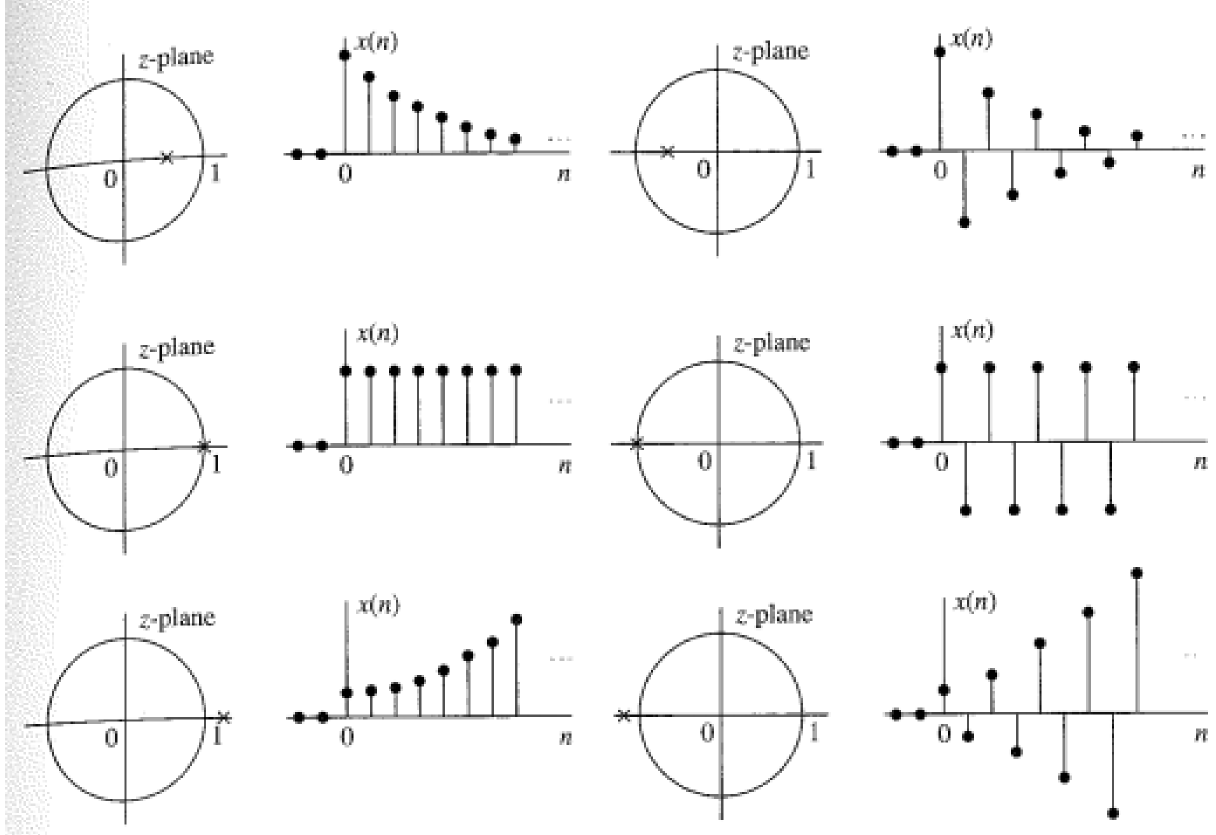

Figure 3.3.5 Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

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Pole Location and Time Domain Behaviour of Causal Signals

$$x[n] = na^n u[n]$$

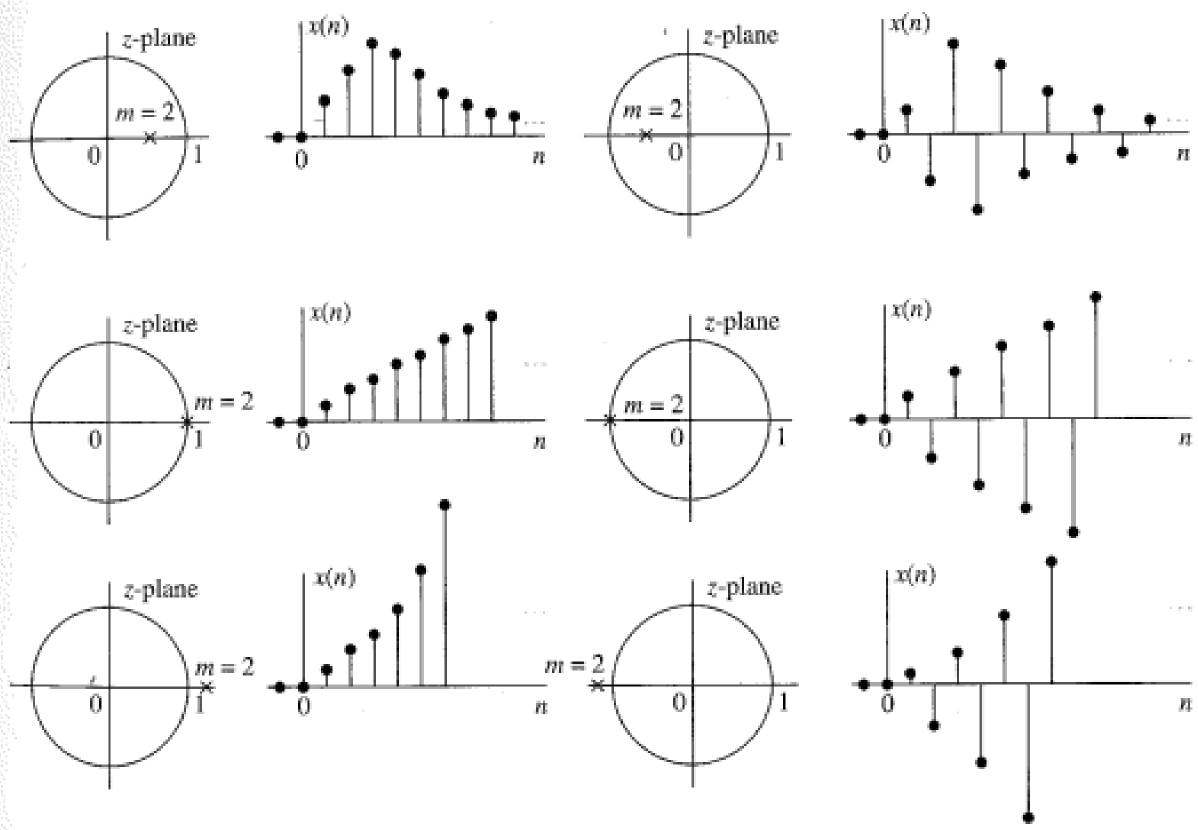

Figure 3.3.6 Time-domain behavior of causal signals corresponding to a double ($m = 2$) real pole, as a function of the pole location.

Pole Location and Time Domain Behaviour of Causal Signals

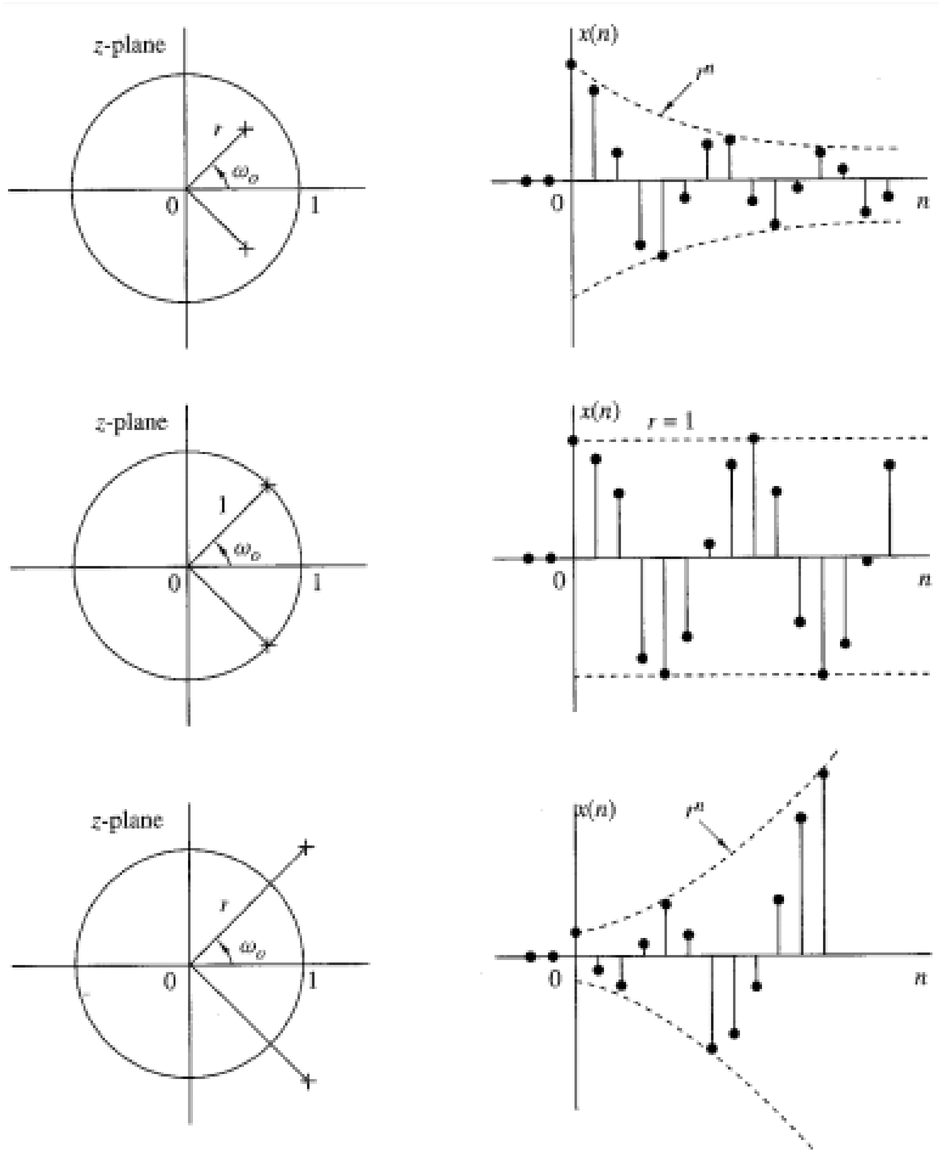

Figure 3.3.7 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

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Pole Location and Time Domain Behaviour of Causal Signals

Summary:

- Causal real signals with simple real poles or simple complex conjugate pairs of poles, which are inside or on the unit circle, are always bounded in amplitude.
- A signal with a pole near the origin decays more rapidly than one associated with a pole near (but not outside) the unit circle.

Suggested Reading

Section 3.0 – 3.2 (Oppenheim)

Practice Problems

Problems: 3.1, 3.3, 3.4, 3.7, 3.10 – 3.12, 3.16 – 3.20 (Oppenheim)