EEE 324 Digital Signal Processing

Lecture 9

Inverse z-Transform

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The Inverse z-Transform

1. Inspection Method

- Recognize “by inspection” certain transform pairs.
- For example, we know that for a right sided exponential sequence:

$$a^n u[n] \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

- So if we need to find the inverse transform of:

$$\frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

We can recall the z-transform pair above and can find the inverse z- transform “by inspection” as:

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

The Inverse z-Transform

1. Inspection Method

- Similarly, we know that for a left sided exponential sequence:

$$-a^n u[-n-1] \overset{z}{\leftrightarrow} \frac{1}{1-az^{-1}}, \qquad |z| < |a|$$

- So if we need to find the inverse transform of:

$$\frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}$$

We can recall the z-transform pair above and can find the inverse z-transform “by inspection” as:

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

The Inverse z-Transform

2. Partial Fraction Expansion Method

- Sometimes $X(z)$ may not be given explicitly in an available table, but it may be possible to obtain an alternative expression for $X(z)$ as a sum of simpler terms each of which is available in a table.
- Particularly useful in the case of rational functions.

The Inverse z-Transform

2. Partial Fraction Expansion Method

$X(z)$ is generally expressed as a ratio of polynomials in z^{-1} or z; i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad (5)$$

Or

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} \qquad (6)$$

- Eq. (6) clearly shows that for such functions, there will be M zeros and N poles.
- Z-transforms of the form of Eq. (5) always have the same number of poles and zeros in the finite z-plane and there are no poles or zeros at infinity.
- In addition, there will be
  - $M - N$ poles at $z = 0$ if $M > N$, or
  - $N - M$ zeros at $z = 0$ if $N > M$

The Inverse z-Transform

2. Partial Fraction Expansion Method

- To obtain the PFE of $X(z)$ in Eq. (5), $X(z)$ is first expressed in the form:

$$X(z) = \frac{b_0 \prod_{k=1}^{M}(1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})} \qquad (7)$$

Where

- the c_k's are the non-zero zeros of $X(z)$ and
- the d_k's are the non-zero poles of $X(z)$.

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2. Partial Fraction Expansion Method

- If $\boldsymbol{M} < \boldsymbol{N}$ and the poles are all first order, then $X(z)$ can be expressed as:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad (8)$$

- Multiplying both sides of Eq. (8) by $(1 - d_k z^{-1})$ and evaluation for $z = d_k$ shows that the coefficients A_k can be found from:

$$A_k = (1 - d_k z^{-1}) X(z)|_{z=d_k} \qquad (9)$$

The Inverse z-Transform

2. Partial Fraction Expansion Method

Example 3.8 (Second-order z-transform)

The Inverse z-Transform

2. Partial Fraction Expansion Method

- If $\boldsymbol{M \geq N}$, then Eq. (8) needs to be modified as:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \qquad (10)$$

  - The B_r's can be obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator.
  - The A_k's can be obtained as in Eq. (9).

The Inverse z-Transform

2. Partial Fraction Expansion Method

- If $\boldsymbol{M \geq N}$, and $X(z)$ has multiple order poles, then Eq. (8) needs to be further modified.
- In particular, if $X(z)$ has a pole of order s at $z = d_i$ and all the other poles are first-order, then Eq. (8) becomes:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m} \quad (11)$$

  - The B_r's and A_k's are obtained as before.
  - The C_m's are obtained as:

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[(1 - d_i w)^s X(w^{-1}) \right] \right\}_{w = d_i^{-1}} \quad (12)$$

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The Inverse z-Transform

2. Partial Fraction Expansion Method

Example 3.9 (Inverse by PFE)

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The Inverse z-Transform

3. Power Series Expansion Method

- If the z-transform is given as a power series in the form:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots,$$

- We can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}.
- This approach is very useful for finite-length sequences where $X(z)$ may have a no simpler form than a polynomial in z^{-1}.

The Inverse z-Transform

3. Power Series Expansion Method

Example 10 (Finite-length Sequence)

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The Inverse z-Transform

3. Power Series Expansion Method

Example 11 (Inverse Transform by PSE)

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad \text{for } |x| < 1$$

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3. Power Series Expansion Method

Example 12 (PSE by Long Division)

The Inverse z-Transform

3. Power Series Expansion Method

Example 13 (PSE for a left-sided sequence)

Suggested Reading

<u>Section 3.3 (Oppenheim)</u>

Practice Problems

Practice Problems: 3.5, 3.6, 3.7, 3.14, 3.15 (Oppenheim)